

UNIT 1 – BASIC PROBABILITY THEORY**❖ INTRODUCTION**

- ✓ Probability theory is the branch of mathematics that is concerned with random (or chance) phenomena. It has attracted people to its study both because of its intrinsic interest and its successful applications to many areas within the physical, biological, social sciences, in engineering and in the business world.
- ✓ The words PROBABLE and POSSIBLE CHANCES are quite familiar to us. We use these words when we are sure of the result of certain events. These words convey the sense of uncertainty of occurrence of events.
- ✓ Probability is the word we use to calculate the degree of the certainty of events.
- ✓ There are two types of approaches in the theory of Probability.
- ✓ Classical Approach – By Blaise Pascal
- ✓ Axiomatic Approach – By A. Kolmogorov

❖ RANDOM EXPERIMENT

- ✓ Random experiment is an experiment about whom outcomes cannot be successfully predicted. Of course, we know all possible outcomes in advance.

❖ SAMPLE SPACE

- ✓ The set of all possible outcomes of a random experiment is called a sample space.
- ✓ It is denoted by “S” and if a sample space is in one-one correspondence with a finite set, then it is called a finite sample space. Otherwise it is known as an infinite sample space.
- ✓ Examples:
- ✓ Finite Sample Space: Experiment of tossing a coin twice.

$$S = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$$

- ✓ Infinite Sample Space: Experiment of tossing a coin until a head comes up for first time.

$$S = \{H, TH, TTH, TTTH, TTTTH, TTTTTH, \dots\}$$

❖ EVENT

- ✓ A subset of a sample space is known as Event. Each member is called Sample Point.
- ✓ Example:

- Experiment: Tossing a coin twice. $S = \{HH, HT, TH, TT\}$
- Event A: Getting TAIL both times. $A = \{TT\}$
- Event B: Getting TAIL exactly once. $B = \{HT, TH\}$

❖ DEFINITIONS

- ✓ The subset \emptyset of a sample space is called “Impossible Events”.
- ✓ The subset S (itself) of a sample space is called “Sure/Certain Events”.
- ✓ If Subset contains only one element, it is called “Elementary/Simple Events”.
- ✓ If Subset contains more than one element, it is called “Compound/Decomposable Events”.
- ✓ A set contains all elements other than A is called “Complementary Event” of A . It is denoted by A' .
- ✓ A Union of Events A and B is Union of sets A and B (As per set theory).
- ✓ An Intersection of Events A and B is Intersection of sets A and B (As per set theory).
- ✓ If $A \cap B = \phi$. Events are called Mutually Exclusive Events (Disjoint set).
 - Set Notation: $A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$
- ✓ If $A \cup B = S$. Events are called Mutually Exhaustive Events.
 - Set Notation: $A \cup B = \{x \mid x \in A \text{ OR } x \in B\}$
- ✓ If $A \cap B = \phi$ and $A \cup B = S$. Events are called Mutually Exclusive & Exhaustive Events.

METHOD – 1: BASIC EXAMPLES ON SAMPLE SPACE AND EVENT

H	1	Define Mutually Exclusive and Exhaustive events with a suitable example.	
C	2	A coin is tossed twice, and their up faces are recorded. What is the sample space for this experiment? Answer: $S = \{HH, HT, TH, TT\}$	
C	3	Suppose a pair of dice are tossed. What is the sample space for the experiment? Answer: $S = \{(1, 1), (1, 2), \dots (1, 6), \dots \dots \dots, (6, 1), (6, 2), \dots (6, 6)\}$	
H	4	Four card are labeled with A, B, C and D. We select two cards at random without replacement. Describe the sample space for the experiments.	

		Answer: $S = \{AB, AC, AD, BC, BD, CD\}$	
H	5	Describe the sample space for the indicated random experiments. (a) A coin is tossed 3 times. (b) A coin and die is tossed together. Answer: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$	
H	6	A balanced coin is tossed thrice. If three tails are obtained, a balance die is rolled. Otherwise the experiment is terminated. Write down the elements of the sample space. Answer: $S = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT1, TTT2, TTT3, TTT4, TTT5, TTT6\}$	
C	7	Two unbiased dice are thrown. Write down the following events: Event A: Both the dice show the same number. Event B: The total of the numbers on the dice is 8. Event C: The total of the numbers on the dice is 13. Event D: The total of the number on the dice is any number from [2, 12]. Answer: $A = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \}$ $B = \{ (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) \}$ $C = \{\emptyset\}$ $D = \{ (1, 1), \dots, (1, 6), \dots, (6, 1), \dots, (6, 6) \}$	
H	8	Let a coin be tossed. If it shows head we draw a ball from a box containing 3 identical red and 4 identical green balls and if it shows a tail, we throw a die. What is the sample space of experiments? Answer: $S = \{HR_1, HR_2, HR_3, HG_1, HG_2, HG_3, HG_4, T1, T2, T3, T4, T5, T6\}$	
H	9	A coin is tossed 3 times. Give the elements of the following events: Event A: Getting at least two heads Event B: Getting exactly two tails Event C: Getting at most one tail Event D: Getting at least one tail Answer: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ EVENT A = $\{HHH, HHT, HTH, THH\}$ EVENT B = $\{HTT, THT, TTH\}$ EVENT C = $\{HHH, HHT, HTH, THH\}$ EVENT D = $\{HHT, HTH, HTT, THH, THT, TTH, TTT\}$	

❖ **PROBABILITY OF AN EVENT**

- ✓ If a finite sample space associated with a random experiments has "n" equally likely (Equiprobable) outcomes (elements) and of these "m" ($0 \leq m \leq n$) outcomes are favorable for the occurrence of an event A, then probability of A is defined as follow

$$P(A) = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{m}{n}$$

❖ **EQUIPROBABLE EVENTS**

- ✓ Let $U = \{x_1, x_2, \dots, x_n\}$ be a finite sample space. If $P\{x_1\} = P\{x_2\} = P\{x_3\} = \dots = P\{x_n\}$, then the elementary events $\{x_1\}, \{x_2\}, \{x_3\}, \dots, \{x_n\}$ are called Equiprobable Events.

❖ **RESULTS**

- ✓ For the Impossible Event $P(\phi) = 0$.
- ✓ Complementation Rule: For every Event A, $P(A') = 1 - P(A)$.
- ✓ If $A \subset B$, than $P(B - A) = P(B) - P(A)$ and $P(A) \leq P(B)$.
- ✓ For every event A, $0 \leq P(A) \leq 1$.
- ✓ Let S be sample space and A, B and C be any events in S, then
 - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 - $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
 - $P(A \cap B') = P(A) - P(A \cap B)$
 - $P(A' \cap B) = P(B) - P(A \cap B)$
 - $P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$ (De Morgan's Rule)
 - $P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B)$ (De Morgan's Rule)

❖ **PERMUTATION**

- ✓ Suppose that we are given 'n' distinct objects and wish to arrange 'r' of these objects in a line. Since there are 'n' ways of choosing the 1st object, after this is done 'n-1' ways of choosing the 2nd object and finally n-r+1 ways of choosing the rth object, it follows by the fundamental principle of counting that the number of different arrangement (or PERMUTATIONS) is given by

$${}^n P_r = n(n-1)(n-2) \dots \dots \dots (n-r+1) = \frac{n!}{(n-r)!}$$

❖ **RESULTS ON PERMUTATION**

- ✓ Suppose that a set consists of 'n' objects of which n_1 are of one type, n_2 are of second type,, and n_k are of k^{th} type. Here $n = n_1 + n_2 + \dots + n_k$. Then the number of different permutations of the objects is

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

- A number of different permutations of letters of the word MISSISSIPPI is

$$\frac{11!}{1! 4! 4! 2!} = 34650$$

- ✓ If 'r' objects are to be arranged out of 'n' objects and if repetition of an object is allowed then the total number of permutations is n^r .
- Different numbers of three digits can be formed from the digits 4, 5, 6, 7, 8 is $5^3 = 125$.

❖ **COMBINATION**

- ✓ In a permutation we are interested in the order of arrangement of the objects. For example, ABC is a different permutation from BCA. In many problems, however, we are interested only in selecting or choosing objects without regard to order. Such selections are called combination.
- ✓ The total number of combination (selections) of 'r' objects selected from 'n' objects is denoted and defined by

$${}^n C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

❖ **EXAMPLES ON COMBINATION**

- ✓ The number of ways in which 3 card can be chosen from 8 cards is

$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

- ✓ A club has 10 male and 8 female members. A committee composed of 3 men and 4 women is formed. In how many ways this be done?

$$\binom{10}{3} \binom{8}{4} = 120 \times 70 = 8400$$

- ✓ Out of 6 boys and 4 girls in how many ways a committee of five members can be formed in which there are at most 2 girls are included?

$$\binom{4}{2}\binom{6}{3} + \binom{4}{1}\binom{6}{4} + \binom{4}{0}\binom{6}{5} = 120 + 60 + 6 = 186$$

METHOD – 2: EXAMPLES ON PROBABILITY OF EVENTS

C	1	If probability of event A is $\frac{9}{10}$, what is the probability of the event “not A”? Answer: 0.1	
C	2	What is the probability that a leap year contains 53 Sundays? Answer: 0.2857	
H	3	Three coins are tossed. Find the probability of (a) Getting at least 2 heads, (b) Getting exactly 2 head. Answer: 0.5, 0.375	
H	4	A single die is tossed once. Find the probability of a 2 or 5 turning up. Answer: $\frac{1}{3}$	
H	5	Two unbiased dice are thrown. Find the probability that: (a) Both the dice show the same number. (b) The first die shows 6. (c) The total of the numbers on the dice is 8. (d) The total of the numbers on the dice is greater than 8. (e) The total of the numbers on the dice is 13. (f) Total of numbers on the dice is any number from 2 to 12, both inclusive. Answer: $\frac{1}{6}, \frac{1}{6}, \frac{5}{36}, \frac{5}{18}, 0, 1$	
C	6	(a) A club has 5 male and 7 female members. A committee composed of 3 men and 4 women is formed. In how many ways this be done? (b) Out of 6 boys and 4 girls in how many ways a committee of five members can be formed in which there are at most 2 girls are included? Answer: 350, 186	

H	7	One card is drawn at random from a well shuffled pack of 52 cards. Find probability that the card will be (a) an ace, (b) a card of black color, (c) a diamond, (d) not an ace. Answer: 0.0769, 0.5, 0.25, 0.9231	
C	8	In a game of poker 5 cards are drawn from a pack of 52 well-shuffled cards. Find the probability of (a) 4 ace, (b) 4 aces and 1 is a king, (c) 3 are tens and 2 are jacks, (d) a nine, ten, jack, queen, king is obtained in any order, (e) 3 are of any one suit and 2 are of another, (f) at least one ace is obtained. Answer: $\frac{1}{54145}$, $\frac{1}{649740}$, $\frac{1}{108290}$, $\frac{64}{162435}$, $\frac{429}{4165}$, $\frac{18472}{54145}$	
H	9	Four cards are drawn from the pack of cards. Find the probability that (a) all are diamonds, (b) there is one card of each suit, (c) there are two spades and two hearts. Answer: 0.0026, 0.1055, 0.0225	
T	10	Consider a poker hand of five cards. Find the probability of getting four of a kind (i.e., four cards of the same face value) assuming the five cards are chosen at random. Answer: $\frac{1}{4165}$	
H	11	4 cards are drawn at random from a pack of 52 cards. Find probability that (a) They are a king, a queen, a jack and an ace. (b) Two are kings and two are queens. (c) Two are black and two are red. (d) There are two cards of hearts and two cards of diamonds. Answer: 0.00095, 0.00013, 0.3902, 0.0225	

H	12	A box contains 5 red, 6 white and 2 black balls. The balls are identical in all respect other than color (a) one ball is drawn at random from the box. Find the probability that the selected ball is black, (b) two balls are drawn at random from the box. Find the probability that one ball is white and one is red. Answer: $\frac{2}{13}, \frac{5}{13}$	
H	13	There are 5 yellow, 2 red, and 3 white balls in the box. Three balls are randomly selected from the box. Find the probability of the following events. (a) all are of different color, (b) 2 yellow and 1 red color, (c) all are of same color. Answer: 0.25, 0.1667, 0.0917	
C	14	An urn contains 6 green, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that at least one is green. Answer: $\frac{683}{969}$	
T	15	A box contains 6 red balls, 4 white balls, 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each color. Answer: 0.5275	
T	16	A machine produces a total of 12000 bolts a day, which are on the average 3% defective. Find the probability that out 600 bolts chosen at random, 12 will be defective. Answer: $\frac{\binom{360}{12} \binom{11640}{588}}{\binom{12000}{600}}$	
C	17	If 3 of 20 tires in storage are defective and 4 of them are randomly chosen for inspection (that is, each tire has the same chance of being selected), what is the probability that the one of the defective tire will be included? Answer: 0.4211	W 2019 (4)

C	18	<p>A room has three lamp sockets. From a collection of 10 light bulbs of which only 6 are good. A person selects 3 at random and puts them in the socket. What is the probability that the room will have light?</p> <p>Answer: $\frac{29}{30}$</p>	
H	19	<p>Do as directed:</p> <p>(a) Find the probability that there will be 5 Sundays in the month of July.</p> <p>(b) Find the probability that there will be 5 Sundays in the month of June.</p> <p>(c) What is the probability that a non-leap year contains 53 Sundays?</p> <p>(d) What is the probability that a leap year contains 53 Sundays?</p> <p>Answer: $\frac{3}{7}, \frac{2}{7}, \frac{1}{7}, \frac{2}{7}$</p>	
H	20	<p>If A and B are two mutually exclusive events with $P(A) = 0.30, P(B) = 0.45$. Find the probability of $A', A \cap B, A \cup B, A' \cap B'$.</p> <p>Answer: 0.7, 0, 0.75, 0.25</p>	
H	21	<p>Let A and B be two events with $P(A) = \frac{2}{3}, P(A \cap B) = \frac{14}{45}, P(A \cup B) = \frac{4}{5}$ then find $P(B)$.</p> <p>Answer: $\frac{4}{9}$</p>	
C	22	<p>The probability that a student passes a physics test is $\frac{2}{3}$ and the probability that he passes both physics and English tests is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$, what is the probability that he passes the English test?</p> <p>Answer: $\frac{4}{9}$</p>	
H	23	<p>A basket contains 20 apples and 10 oranges of which 5 apples and 3 oranges are bad. If a person takes 2 at random, what is the probability that either both are apples or both are good?</p> <p>Answer: $\frac{316}{435}$</p>	

C	24	Two dice are thrown together. Find the probability that the sum is divisible by 2 or 3. Answer: 0.6667	
H	25	A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card. Answer: $\frac{7}{13}$	
H	26	An integer is chosen at random from the first 200 positive integers. What is the probability that the integer is divisible by 6 or 8? Answer: 0.25	
T	27	Three newspapers A, B, C are published in a certain city. It is estimated from a survey that of the adult population: 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read B and C, 2% read all three. Find what percentage read at least one of the papers? Answer: 35%	
T	28	Four letters of the words THURSDAY are arranged in all possible ways. Find the probability that the word formed is HURT. Answer: $\frac{1}{1680}$	
H	29	A class has 10 boys and 5 girls. Three students are selected at random one after the other. Find the probability that (a) First two are boys and third is girls. (b) First and third of same gender and second is of opposite gender. Answer: $\frac{15}{91}$, $\frac{5}{21}$	
H	30	In how many different ways can 4 of 15 laboratory assistants be chosen to assist with an experiment? Answer: 1365	W 2019 (3)

❖ **CONDITIONAL PROBABILITY**

- ✓ Let S be a sample space and A and B be any two events in S . Then the probability of the occurrence of event A when it is given that B has already occurred is define as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}; P(B) > 0$$

- ✓ Which is known as conditional probability of the event A relative to event B .
- ✓ Similarly, the conditional probability of the event B relative to event A is

$$P(B/A) = \frac{P(B \cap A)}{P(A)}; P(A) > 0$$

- ✓ Properties:

- Let A_1, A_2 and B be any three events of a sample space S , then

$$P(A_1 \cup A_2/B) = P(A_1/B) + P(A_2/B) - P(A_1 \cap A_2/B); P(B) > 0.$$

- Let A and B be any two events of a sample space S , then

$$P(A'/B) = 1 - P(A/B); P(B) > 0.$$

❖ **THEOREM (MULTIPLICATION RULE)**

- ✓ Let S be a sample space and A and B be any two events in S , then

$$P(A \cap B) = P(A) \cdot P(B/A); P(A) > 0 \text{ or } P(A \cap B) = P(B) \cdot P(A/B); P(B) > 0.$$

- ✓ Let S be a sample space and A, B and C be three events in S , then

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$$

❖ **INDEPENDENT EVENTS**

- ✓ Let A and B be any two events of a sample space S , then A and B are called independent events if $P(A \cap B) = P(A) \cdot P(B)$.
- ✓ It also means that, $P(A/B) = P(A)$ and $P(B/A) = P(B)$.
- ✓ This means that the probability of A does not depend on the occurrence or nonoccurrence of B , and conversely.

❖ **REMARKS**

- ✓ Let A, B and C are said to be Mutually independent, if
 - $P(A \cap B) = P(A) \cdot P(B)$ and $P(B \cap C) = P(B) \cdot P(C)$

- $P(C \cap A) = P(C) \cdot P(A)$ and $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$
- ✓ Let A, B and C are said to be Pairwise independent, if
 - $P(A \cap B) = P(A) \cdot P(B)$
 - $P(B \cap C) = P(B) \cdot P(C)$
 - $P(C \cap A) = P(C) \cdot P(A)$

METHOD - 3: EXAMPLES ON CONDITIONAL PROBABILITY

C	1	If $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$ and $P(A \cup B) = \frac{11}{12}$. Find $P(A/B)$. Answer: $\frac{2}{9}$	
H	2	If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cup B) = \frac{1}{2}$, then find $P(B/A)$, $P(A/B')$. Answer: $\frac{1}{4}$, $\frac{1}{3}$	
C	3	$P(A) = \frac{1}{3}$, $P(B') = \frac{1}{4}$, $P(A \cap B) = \frac{1}{6}$, then find $P(A \cup B)$, $P(A' \cap B')$ and $P(A'/B')$. Answer: $\frac{11}{12}$, $\frac{1}{12}$, $\frac{1}{3}$	
C	4	A card is drawn from a well-shuffled deck of 52 cards and then second card is drawn, find the probability that one card is a spade and then second card is club if the first card is not replaced. Answer: $\frac{13}{204}$	
H	5	In a group of 200 students 40 are taking English, 50 are taking math, 12 are taking both. (a) if a student is selected at random, what is the probability that the student is taking English? (b) a student is selected at random from those taking math. What is the probability that the student is taking English? (c) a student is selected at random from those taking English, what is the probability that the student is taking math? Answer: 0.20 , 0.24 , 0.3	

H	6	<p>In a box, 100 bulbs are supplied out of which 10 bulbs have defects of type A, 5 bulbs have defects of type B and 2 bulbs have defects of both the type. Find the probability that (a) a bulb to be drawn at random has a B type defect under the condition that it has an A type defect, (b) a bulb to be drawn at random has no B type defect under the condition that it has no A type defect.</p> <p>Answer: 0.2, 0.9667</p>
C	7	<p>In a certain college 25% of the students failed in probability and 15% of the student failed in statistics. A student is selected at random and 10% of the students failed in both. If he failed in probability, what is probability that he failed in statistics?</p> <p>Answer: 0.4</p>
H	8	<p>Two integers are selected at random from 1 to 11. If the sum is even, find the probability that both the integers are odd.</p> <p>Answer: 0.6</p>
C	9	<p>From a bag containing 4 white and 6 black balls, two balls are drawn at random. If the balls are drawn one after the other without replacements, find the probability that one is white and one is black.</p> <p>Answer: $\frac{4}{15}$</p>
C	10	<p>In producing screws, let A mean “screw too slim” and B “screw too short”. Let $p(A) = 0.1$ and $P(B \cap A) = 0.02$. A screw, selected randomly, is of type A, what is probability that a screw is of type B.</p> <p>Answer: 0.2</p>
H	11	<p>A bag contains 6 white, 9 black balls. 4 balls are drawn at a time. Find the probability for first draw to give 4 white & second draw to give 4 black balls in each of following cases.</p> <p>(a) The balls are replaced before the second draw.</p> <p>(b) The balls are not replaced before the second draw.</p> <p>Answer: $\frac{6}{5915}$, $\frac{3}{715}$</p>

H	12	For two independent events A & B if $P(A) = 0.3$, $P(A \cup B) = 0.6$, find $P(B)$. Answer: 0.4286	
H	13	If A, B are independent events and $P(A) = 1/4$, $P(B) = 2/3$. Find $P(A \cup B)$. Answer: 0.75	
C	14	If A and B are independent events, with $P(A) = 3/8$, $P(B) = 7/8$. Find $P(A \cup B)$, $P(A/B)$ and $P(B/A)$. Answer: $\frac{59}{64}$, $\frac{3}{8}$, $\frac{7}{8}$	
C	15	Let S be square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in plane. Consider the uniform probability space on square. Show that A and B are independent events if $A = \{(x,y): 0 \leq x \leq \frac{1}{2}, 0 \leq y \leq 1\}$ & $B = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{4}\}$.	
H	16	A person is known to hit the target in 3 out of 4 shots, whereas another person is known to hit the target in 2 out of 3 shots. What is probability that target will be hit? Answer: $\frac{11}{12}$	
T	17	A problem in statistics is given to three students A, B, C whose chances of solving it are 0.5, 0.75 and 0.25 respectively. What is the probability that the problem will be solved if all of them try independently? Answer: $\frac{29}{32}$	
H	18	If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls. What is the probability that one of the balls is white and the other two black? Answer: 0.3636	W 2019 (3)

❖ TOTAL PROBABILITY

- ✓ If B_1 & B_2 are two mutually exclusive and exhaustive events of sample space S and $P(B_1), P(B_2) \neq 0$, then for any event A,

$$P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)$$

- ✓ If B_1, B_2 and B_3 are mutually exclusive and exhaustive events and $P(B_1), P(B_2), P(B_3) \neq 0$, then for any event A.

$$P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) + P(B_3) \cdot P(A/B_3)$$

❖ BAYES' THEOREM

- ✓ Let $B_1, B_2, B_3, \dots, B_n$ be n-mutually exclusive and exhaustive events of a sample space S and let A be any event such that $P(A) \neq 0$, then

$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) + \dots + P(B_n) \cdot P(A/B_n)}$$

METHOD – 4: EXAMPLES ON TOTAL PROBABILITY AND BAYES' THEOREM

C	1	Consider two boxes, first with 5-green & 2-pink and second with 4-green & 3-pink balls. Two balls are selected from random box. If both balls are pink, find the probability that they are from second box. Answer: $\frac{3}{4}$	
H	2	In a certain assembly plant, three machines, B_1, B_2 and B_3 , make 30%, 45% and 25%, respectively, of the products. It is known from the past experience that 2%, 3% and 2% of the products made by each machine respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective? Answer: 0.0245	
H	3	There are three boxes. Box I contains 10 light bulbs of which 4 are defective. Box II contains 6 light bulbs of which 1 is defective and box III contains 8 light bulbs of which 3 are defective. A box is chosen and a bulb is drawn. Find the probability that the bulb is defective. Answer: 0.3139	

T	4	An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the later. What is the probability that it is a white ball? Answer: $\frac{59}{130}$	
C	5	Suppose that the population of a certain city is 40% male & 60% female. Suppose also that 50% of males & 30% of females smoke. Find the probability that a smoker is male. Answer: $\frac{10}{19}$	
H	6	A microchip company has two machines that produce the chips. Machine-I produces 65% of the chips, but 5% of its chips are defective. Machine-II produces 35% of the chips, but 15% of its chips are defective. A chip is selected at random and found to be defective. What is the probability that it came from Machine-I? Answer: 0.3824	W 2019 (4)
H	7	State Bayes' theorem. In a bolt factory, three machines A, B and C manufacture 25%, 35% and 40% of the total product respectively. Out of these outputs 5%, 4% and 2% respectively, are defective bolts. A bolt is picked up at random and found to be defective. What are the Probabilities that it was manufactured by machine A, B and C? Answer: 0.3623, 0.4057, 0.2318	
H	8	A company has two plants to manufacture hydraulic machine. Plat I manufacture 70% of the hydraulic machines and plant II manufactures 30%. At plant I, 80% of hydraulic machines are rated standard quality and at plant II, 90% of hydraulic machine are rated standard quality. A machine is picked up at random and is found to be of standard quality. What is the chance that it has come from plant I? Answer: 0.6747	

H	9	<p>There are two boxes A and B containing 4 white, 3 red and 3 white, 7 red balls respectively. A box is chosen at random and a ball is drawn from it, if the ball is white, find the probability that it is from box A.</p> <p>Answer: $\frac{40}{61}$</p>	
H	10	<p>Urn A contain 1 white, 2 black, 3 red balls; Urn B contain 2 white, 1 black, 1 red balls; Urn C contain 4 white, 5 black, 3 red balls. One urn is chosen at random & two balls are drawn. These happen to be one white & one red. What is probability that they come from urn A?</p> <p>Answer: 0.2268</p>	
C	11	<p>Three hospitals contain 10%, 20% and 30% of diabetes patients. A Patient is selected at random who is diabetes patient. Determine the probability that this patient comes from first hospital.</p> <p>Answer: 0.1667</p>	
C	12	<p>In a computer engineering class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of student are boys. If a student is selected random and found to have IQ more than 150, find the probability that the student is a boy.</p> <p>Answer: $\frac{3}{7}$</p>	
H	13	<p>An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver and truck driver is 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident, what is the probability that he is a scooter driver?</p> <p>Answer: $\frac{1}{52}$</p>	

H	14	<p>A factory has three machine X, Y, Z producing 1000, 2000, 3000 bolts per day respectively. Machine X produces 1% defective bolts, Y produces 1.5%, Z produces 2% defective bolts. At end of the day, a bolt is drawn at random and it is found to be defective. What is the probability that this defective bolt has been produced by the machine X?</p> <p>Answer: 0.1</p>	
T	15	<p>A card from a pack of 52 cards is lost. From the remaining cards of pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.</p> <p>Answer: $\frac{11}{50}$</p>	
T	16	<p>Suppose there are three chests each having two drawers. The first chest has a gold coin in each drawer, the second chest has a gold coin in one drawer and a silver coin in the other drawer and the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer opened. If the drawer contains a gold coin, what is the probability that the other drawer also contains a gold coin?</p> <p>Answer: $\frac{2}{3}$</p>	
T	17	<p>If proposed medical screening on a population, the probability that the test correctly identifies someone with illness as positive is 0.99 and the probability that test correctly identifies someone without illness as negative is 0.95. The incident of illness in general population is 0.0001. You take the test the result is positive then what is the probability that you have illness?</p> <p>Answer: 0.002</p>	

❖ RANDOM VARIABLE

- ✓ A random variable is a variable whose value is unknown or a function that assigns values to each of an experiment's outcomes. Random variables are often designated by capital letters X, Y.

✓ Random variables can be classified as

1) Discrete Random variables, which are variable that have specific values.

2) Continuous Random variables, which are variables that can have any values within a continuous range.

❖ PROBABILITY DISTRIBUTION OF RANDOM VARIABLE

✓ Probability distribution of random variable is the set of its possible values together with their respective probabilities. It means,

X	x_1	x_2	x_3	x_n
P(X)	$p(x_1)$	$p(x_2)$	$p(x_3)$	$p(x_n)$

where $p(x_i) \geq 0$ and $\sum_i p(x_i) = 1$ for all i.

✓ **Example:** Two balanced coins are tossed, find the probability distribution for heads.

➤ Sample space = {HH, HT, TH, TT}

➤ $P(X = 0) = P(\text{no head}) = \frac{1}{4} = 0.25$ and $P(X = 1) = P(\text{one head}) = \frac{2}{4} = \frac{1}{2} = 0.5$

➤ $P(X = 2) = P(\text{two heads}) = \frac{1}{4} = 0.25$

➤ Probability distribution is as follow:

X	0	1	2
P(X=x)	0.25	0.5	0.25

❖ DISCRETE RANDOM VARIABLE

✓ A random variable, which can take only finite, countable, or isolated values in a given interval, is called discrete random variable.

✓ A random variable is one, which can assume any of a set of possible values which can be counted or listed.

✓ A discrete random variable is a random variable with a finite (or countably infinite) range.

✓ For example, the numbers of heads in tossing 2 coins.

✓ Discrete random variables can be measured exactly.

❖ CONTINUOUS RANDOM VARIABLE

- ✓ A random variable, which can take all possible values that are infinite in a given interval, is called Continuous random variable.
- ✓ A continuous random variable is one, which can assume any of infinite spectrum of different values across an interval which cannot be counted or listed.
- ✓ For example, measuring the height of a student selected at random.
- ✓ Continuous random variables cannot be measured exactly.

❖ PROBABILITY FUNCTION

- ✓ If for random variable X , the real valued function $f(x)$ is such that $P(X = x) = f(x)$, then $f(x)$ is called Probability function of random variable X .
- ✓ Probability function $f(x)$ gives the measures of probability for different values of X say x_1, x_2, \dots, x_n .
- ✓ Probability functions can be classified as (1) Probability Mass Function (P. M. F.) or (2) Probability Density Function (P. D. F.).

❖ PROBABILITY MASS FUNCTION

- ✓ If X is a discrete random variable then its probability function $P(X)$ is discrete probability function. It is also called probability mass function.
- ✓ Conditions:
 - $p(x_i) \geq 0$ for all i
 - $\sum_{i=1}^n p(x_i) = 1$

❖ PROBABILITY DENSITY FUNCTION

- ✓ If X is a continuous random variable then its probability function $f(x)$ is called continuous probability function OR probability density function.
- ✓ Conditions:
 - $f(x_i) \geq 0$ for all i
 - $\int_{-\infty}^{\infty} f(x) dx = 1$
 - $P(a < x < b) = \int_a^b f(x) dx$

❖ **MATHEMATICAL EXPECTATION**

- ✓ If X is a discrete random variable having various possible values x_1, x_2, \dots, x_n & if $P(X)$ is the probability mass function, the mathematical Expectation of X is defined & denoted by

$$E(X) = \sum_{i=1}^n x_i \cdot P(x_i)$$

- ✓ If X is a continuous random variable having probability density function $f(x)$, expectation of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

- ✓ $E(X)$ is also called the mean value of the probability distribution of x and is denoted by μ .

- ✓ Properties:

- Expected value of constant term is constant. i.e. $E(c) = c$
- If c is constant, then $E(cX) = c \cdot E(X)$
- $E(X^2) = \sum_{i=1}^n x_i^2 \cdot P(x_i)$ (PMF)
- $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$ (PDF)
- If a and b are constants, then $E(aX \pm b) = aE(X) \pm b$
- If a, b and c are constants, then $E\left(\frac{aX+b}{c}\right) = \frac{1}{c}[aE(X) + b]$
- If X and Y are two random variables, then $E(X + Y) = E(X) + E(Y)$
- If X and Y are two independent random variable, then $E(X \cdot Y) = E(X) \cdot E(Y)$

❖ **VARIANCE OF A RANDOM VARIABLE:**

- ✓ Variance is a characteristic of random variable X and it is used to measure dispersion (or variation) of X .
- ✓ If X is a discrete random variable (or continuous random variable) with probability mass function $P(X)$ (or probability density function), then expected value of $[X - E(X)]^2$ is called the variance of X and it is denoted by $V(X)$.

$$V(X) = E(X^2) - [E(X)]^2$$

✓ Properties:

- $V(c) = 0$, Where c is a constant
- $V(cX) = c^2 V(X)$, where c is a constant
- $V(X + c) = V(X)$, Where c is a constant
- If a and b are constants, then $V(aX + b) = a^2V(X)$
- If X and Y are the independent random variables, then $V(X + Y) = V(X) + V(Y)$

❖ **STANDARD DEVIATION OF RANDOM VARIABLE**

- ✓ The positive square root of $V(X)$ (Variance of X) is called standard deviation of random variable X and is denoted by σ . i.e. $\sigma = \sqrt{V(X)}$.
- ✓ σ^2 is called variance of $V(X)$.

❖ **CUMULATIVE DISTRIBUTION FUNCTION**

- ✓ An alternate method for describing a random variable's probability distribution is with cumulative probabilities such as $P(X \leq x)$.
- ✓ A Cumulative Distribution Function of a discrete random variable X , denoted as $F(X)$, is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(x_i)$$

- ✓ A Cumulative Distribution Function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt, -\infty < x < \infty$$

- ✓ Example: Determine the probability mass function of X from the following Cumulative distribution function.

$$F(x) = \begin{cases} 0 & ; & x < -2 \\ 0.2 & ; & -2 \leq x < 0 \\ 0.7 & ; & 0 \leq x < 2 \\ 1 & ; & x \geq 2 \end{cases}$$

- **Answer:** The probability mass function of X is,

$$P(-2) = 0.2 - 0 = 0.2, \quad P(0) = 0.7 - 0.2 = 0.5, \quad P(2) = 1 - 0.7 = 0.3.$$

➤ Then probability distribution is:

X	-2	0	2
P(X = x)	0.2	0.5	0.3

✓ Properties:

- $F(-\infty) = 0, F(+\infty) = 1$ and $0 \leq F(x) \leq 1$
- F is a non-decreasing function, i.e. if $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$.
- If $F(x_0) = 0$, then $F(x) = 0$ for every $x \leq x_0$.
- $P\{X > x\} = 1 - F(x)$
- $P(\{x_1 < X \leq x_2\}) = F(x_2) - F(x_1)$

METHOD – 5: EXAMPLES ON RANDOM VARIABLE

C	1	Which of the following functions are probability function? (a) $P(X = x) = \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{1-x}; x = 0,1$ (b) $P(X = x) = \left(-\frac{1}{2}\right)^x; x = 0,1,2$ Answer: yes, no																
H	2	Find expected value of a random variable X having following probability distribution. <table border="1" style="margin: 10px auto;"> <tr> <td>X</td> <td>-5</td> <td>-1</td> <td>0</td> <td>1</td> <td>5</td> <td>8</td> </tr> <tr> <td>P(X = x)</td> <td>0.12</td> <td>0.16</td> <td>0.28</td> <td>0.22</td> <td>0.12</td> <td>0.1</td> </tr> </table> Answer: 0.86	X	-5	-1	0	1	5	8	P(X = x)	0.12	0.16	0.28	0.22	0.12	0.1		
X	-5	-1	0	1	5	8												
P(X = x)	0.12	0.16	0.28	0.22	0.12	0.1												
C	3	The following table gives the probabilities that a certain computer will malfunction 0, 1, 2, 3, 4, 5, or 6 times on any one day. <table border="1" style="margin: 10px auto;"> <tr> <td>No. of malfunction x</td> <td>0</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>6</td> </tr> <tr> <td>Probability f(x)</td> <td>0.17</td> <td>0.29</td> <td>0.27</td> <td>0.16</td> <td>0.07</td> <td>0.03</td> <td>0.01</td> </tr> </table> Find mean and standard deviation of this probability distribution. Answer: 1.8, $\sqrt{1.8}$	No. of malfunction x	0	1	2	3	4	5	6	Probability f(x)	0.17	0.29	0.27	0.16	0.07	0.03	0.01
No. of malfunction x	0	1	2	3	4	5	6											
Probability f(x)	0.17	0.29	0.27	0.16	0.07	0.03	0.01											

H	4	<p>The probability distribution of a random variable x is as follows. Find p and $E(x)$.</p> <table border="1"> <tr> <td>X</td> <td>0</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> </tr> <tr> <td>P(x)</td> <td>p</td> <td>$\frac{1}{5}$</td> <td>$\frac{1}{10}$</td> <td>p</td> <td>$\frac{1}{20}$</td> <td>$\frac{1}{20}$</td> </tr> </table> <p>Answer: $P = 0.30, E(x) = 1.7500$</p>	X	0	1	2	3	4	5	P(x)	p	$\frac{1}{5}$	$\frac{1}{10}$	p	$\frac{1}{20}$	$\frac{1}{20}$				
X	0	1	2	3	4	5														
P(x)	p	$\frac{1}{5}$	$\frac{1}{10}$	p	$\frac{1}{20}$	$\frac{1}{20}$														
C	5	<p>A random variable X has the following function.</p> <table border="1"> <tr> <td>X</td> <td>0</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>6</td> <td>7</td> </tr> <tr> <td>P(X = x)</td> <td>0</td> <td>K</td> <td>2k</td> <td>2K</td> <td>3k</td> <td>k^2</td> <td>$2k^2$</td> <td>$7k^2 + k$</td> </tr> </table> <p>Find the value of k and then evaluate $P(X < 6), P(X \geq 6)$ and $P(0 < x < 5)$.</p> <p>Answer: 0.1, 0.81, 0.19, 0.8</p>	X	0	1	2	3	4	5	6	7	P(X = x)	0	K	2k	2K	3k	k^2	$2k^2$	$7k^2 + k$
X	0	1	2	3	4	5	6	7												
P(X = x)	0	K	2k	2K	3k	k^2	$2k^2$	$7k^2 + k$												
C	6	<p>Probability distribution of a random variable X is given below. Find $E(X), V(X), \sigma(X), E(3X + 2), V(3X + 2)$.</p> <table border="1"> <tr> <td>X</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> </tr> <tr> <td>P(X)</td> <td>0.1</td> <td>0.2</td> <td>0.5</td> <td>0.2</td> </tr> </table> <p>Answer: 2.8, 0.76, 0.8718, 10.4, 6.84</p>	X	1	2	3	4	P(X)	0.1	0.2	0.5	0.2								
X	1	2	3	4																
P(X)	0.1	0.2	0.5	0.2																
H	7	<p>Probability distribution of a random variable X is given below. Find $P(2 \leq x \leq 4)$ and $P(x > 2)$.</p> <table border="1"> <tr> <td>X</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> </tr> <tr> <td>P(X)</td> <td>0.1</td> <td>0.2</td> <td>0.5</td> <td>0.2</td> </tr> </table> <p>Answer: 0.7, 0.7</p>	X	1	2	3	4	P(X)	0.1	0.2	0.5	0.2								
X	1	2	3	4																
P(X)	0.1	0.2	0.5	0.2																
H	8	<p>The probability distribution of a random variable X is given below. Find $a, E(X), E(2X + 3), E(X^2 + 2), V(X), V(3X + 2)$</p> <table border="1"> <tr> <td>X</td> <td>-2</td> <td>-1</td> <td>0</td> <td>1</td> <td>2</td> </tr> <tr> <td>P(x)</td> <td>$\frac{1}{12}$</td> <td>$\frac{1}{3}$</td> <td>a</td> <td>$\frac{1}{4}$</td> <td>$\frac{1}{6}$</td> </tr> </table> <p>Answer: $\frac{1}{6}, \frac{1}{12}, \frac{19}{6}, \frac{43}{12}, \frac{227}{144}, \frac{227}{16}$</p>	X	-2	-1	0	1	2	P(x)	$\frac{1}{12}$	$\frac{1}{3}$	a	$\frac{1}{4}$	$\frac{1}{6}$						
X	-2	-1	0	1	2															
P(x)	$\frac{1}{12}$	$\frac{1}{3}$	a	$\frac{1}{4}$	$\frac{1}{6}$															

H	9	<p>The probability distribution of a random variable X is given below.</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tbody> <tr> <td>X</td> <td>-1</td> <td>0</td> <td>1</td> <td>2</td> <td>3</td> </tr> <tr> <td>P(x)</td> <td>$\frac{3}{10}$</td> <td>$\frac{1}{10}$</td> <td>k</td> <td>$\frac{3}{10}$</td> <td>$\frac{1}{10}$</td> </tr> </tbody> </table> <p>Find k, E(X), E(4X + 3), E(X²), V(X), V(2X + 3).</p> <p>Answer: $\frac{1}{5}, \frac{4}{5}, \frac{31}{5}, \frac{13}{5}, \frac{49}{25}, \frac{196}{25}$</p>	X	-1	0	1	2	3	P(x)	$\frac{3}{10}$	$\frac{1}{10}$	k	$\frac{3}{10}$	$\frac{1}{10}$	
X	-1	0	1	2	3										
P(x)	$\frac{3}{10}$	$\frac{1}{10}$	k	$\frac{3}{10}$	$\frac{1}{10}$										
C	10	<p>If $P(x) = \frac{2x+1}{48}, x = 1, 2, 3, 4, 5, 6$, verify whether p(x) is probability function.</p> <p>Answer: yes</p>													
H	11	<p>If $P(X = x) = \frac{x}{15}, x = 1$ to 5. Find P(1 or 2) & P({0.5 < X < 2.5}/{X > 1}).</p> <p>Answer: 0.1867, 0.1429</p>													
C	12	<p>Find 'k' for the probability distribution $p(x) = k\binom{4}{x}, x = 0, 1, 2, 3, 4$.</p> <p>Answer: $\frac{1}{16}$</p>													
T	13	<p>Let mean and standard deviation of a random variable X be 5 & 5 respectively, find E(X²) and E(2X + 5)².</p> <p>Answer: 50, 325</p>													
C	14	<p>Three balanced coins are tossed, find the mathematical expectation of tails.</p> <p>Answer: 1.5</p>													
T	15	<p>4 raw mangoes are mixed accidentally with the 16 ripe mangoes. Find the probability distribution of the raw mangoes in a draw of 2 mangoes.</p> <p>Answer: $\frac{60}{95}, \frac{32}{95}, \frac{3}{95}$</p>													
C	16	<p>A machine produces on average of 500 items during first week of the month & average of 400 items during the last week of the month. The probability for these being 0.68 and 0.32. Determine the expected value of the production.</p> <p>Answer: 468</p>													

H	17	In a business, the probability that a trader can get profit of Rs. 5000 is 0.4 and probability for loss of Rs. 2000 is 0.6. Find his expected gain or loss. Answer: 800	
C	18	There are 8 apples in a box, of which 2 are rotten. A person selects 3 Apples at random from it. Find the expected value of the rotten Apples. Answer: 0.75	
C	19	There are 3 red and 2 white balls in a box and 2 balls are taken at random from it. A person gets Rs. 20 for each red ball and Rs. 10 for each white ball. Find his expected gain. Answer: 32	
H	20	There are 10 bulbs in a box, out of which 4 are defectives. If 3 bulbs are taken at random, find the expected number of defective bulbs. Answer: 1.2	
C	21	(a) A contestant tosses a coin and receives \$5 if head appears and \$1 if tail appears. What is the expected value of a trial? (b) A contestant receives \$4.00 if a coin turns up heads and pays \$3.00 if it turns tails. What is the expected value of a trail? Answer: \$3.00 , \$0.50	
C	22	Find the constant c such that the function $f(x) = \begin{cases} cx^2 & ; 0 < x < 3 \\ 0 & ; \text{elsewhere} \end{cases}$ is a probability density function and compute $P(1 < X < 2)$. Answer: $\frac{1}{9}, \frac{7}{27}$	
H	23	A random variable X has p. d. f. $f(x) = kx^2(1 - x^3)$; $0 < x < 1$. Find the value of 'k' and hence find its mean and variance. Answer: 6, $\frac{9}{14}, \frac{9}{245}$	
C	24	Check whether $f(x) = \begin{cases} \frac{3+2x}{18} & ; 2 \leq x \leq 4 \\ 0 & ; \text{otherwise} \end{cases}$ is a Probability density function? If yes, then find $P(3 \leq X \leq 4)$. Answer: yes, $\frac{10}{18}$	

H	25	A random variable X has p. d. f. $f(x) = \begin{cases} \frac{3+2x}{18} & ; 2 \leq x \leq 4 \\ 0 & ; \text{otherwise} \end{cases}$. Find the standard deviation of the distribution. Answer: 0.5726	
C	26	A random variable X has p. d. f. $f(x) = kx^2(4-x)$; $0 < x < 4$. Find the value of 'k' and hence find its mean and standard deviation. Answer: 0.0469, 2.40, 0.8	
T	27	For the probability function $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$, find k. Answer: $\frac{1}{\pi}$	
T	28	Verify that the following function is pdf or not: $f(x) = \begin{cases} \frac{x}{8} & ; 0 \leq x < 2 \\ \frac{1}{4} & ; 2 \leq x < 4 \\ \frac{6-x}{8} & ; 4 \leq x < 6 \end{cases}$ Answer: Yes	

❖ TWO DIMENSIONAL RANDOM VARIABLE

- ✓ Let S be the sample space associated with a random experiment E. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcomes. Then (X, Y) is called a two dimensional random variable.

❖ TWO DIMENSIONAL DISCRETE RANDOM VARIABLE

- ✓ If the possible values of (X, Y) are finite or countable infinite, (X, Y) is called a two dimensional discrete random variable.
- ✓ **Example:** Consider the experiment of tossing a coin twice. The sample space $S = \{HH, HT, TH, TT\}$. Let X denotes the number of head obtained in first toss and Y denotes the number of head obtained in second toss. Then

S	HH	HT	TH	TT
X(S)	1	1	0	0
Y(S)	1	0	1	0

✓ Here, (X, Y) is a two-dimensional random variable and the range space of (X, Y) is {(1, 1), (1, 0), (0, 1), (0, 0)} which is finite & so (X, Y) is a two-dimensional discrete random variable. Further,

	Y = 0	Y = 1	Y = 2
X = 0	0.25	0.25	–
X = 1	0.25	0.25	–
X = 2	–	–	–

❖ **TWO DIMENSIONAL CONTINUOUS RANDOM VARIABLE**

✓ If (X, Y) can assume all values in a specified region R in the xy-plane, (X, Y) is called a two dimensional continuous random variable.

❖ **JOINT PROBABILITY MASS FUNCTION (DISCRETE CASE)**

✓ If (X, Y) is a two-dimensional discrete random variable such that $P(X = x_i, Y = y_j) = p_{ij}$, then p_{ij} is called the joint probability mass function of (X, Y) provided $P_{ij} \geq 0$ For all i & j and $\sum_i \sum_j P_{ij} = 1$.

❖ **THE MARGINAL PROBABILITY FUNCTION (DISCRETE CASE)**

✓ The marginal probability function is defined as

$$P_X(x) = \sum_y P(X = x, Y = y) \quad \& \quad P_Y(y) = \sum_x P(X = x, Y = y)$$

✓ **Example:** The joint probability mass function (PMF) of X and Y is

	Y = 0	Y = 1	Y = 2
X = 0	0.1	0.04	0.02
X = 1	0.08	0.2	0.06
X = 2	0.06	0.14	0.3

✓ The marginal probability mass function of X is

➤ $P_X(X = 0) = 0.1 + 0.04 + 0.02 = 0.16$

➤ $P_X(X = 1) = 0.08 + 0.2 + 0.06 = 0.34$

➤ $P_X(X = 2) = 0.06 + 0.14 + 0.3 = 0.5$

✓ The marginal probability mass function of Y is

➤ $P_Y(Y = 0) = 0.1 + 0.08 + 0.06 = 0.24$

➤ $P_Y(Y = 1) = 0.04 + 0.2 + 0.14 = 0.38$

➤ $P_Y(Y = 2) = 0.02 + 0.06 + 0.3 = 0.38$

❖ **JOINT PROBABILITY DENSITY FUNCTION (CONTINUOUS CASE)**

✓ If (X, Y) is a two-dimensional continuous Random Variable, then

$$P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}, y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right) = f(x, y)$$

✓ It is called the joint probability density function of (X, Y), provided $f(x, y) \geq 0$, for all $(x, y) \in D$; Where D is range of space and $\iint_D f(x, y) dx dy = 1$

i. e. $P(a \leq X \leq b, c \leq Y \leq D) = \int_c^d \int_a^b f(x, y) dx dy$

❖ **THE MARGINAL PROBABILITY FUNCTION (CONTINUOUS CASE)**

✓ The marginal probability function is defined as

$$F_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \& \quad F_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

✓ Example: Joint probability density function of two random variables X & Y is given by

$$f(x, y) = \begin{cases} \frac{x^2 - xy}{8} & ; 0 < x < 2 \text{ and } -x < y < x \\ 0 & ; \text{ otherwise} \end{cases}$$

➤ The marginal probability density function of X is

$$F_X(x) = \int_{-x}^x f(x, y) dy = \int_{-x}^x \frac{x^2 - xy}{8} dy = \frac{1}{8} \left(x^2 y - \frac{xy^2}{2} \right)_{-x}^x = \frac{x^3}{4} ; 0 < x < 2$$

➤ The marginal probability density function of Y is

$$F_Y(y) = \int_0^2 f(x, y) dx = \int_0^2 \frac{x^2 - xy}{8} dx = \frac{1}{8} \left(\frac{x^3}{3} - \frac{x^2 y}{2} \right)_0^2 = \frac{1}{3} - \frac{y}{4} ; -x < y < x$$

✓ **Remark:** The marginal distribution function of (X, Y) is

$$F_1(x) = \int_{-\infty}^x \int_{-\infty}^{\infty} f(x,y) \, dx dy \quad \& \quad F_2(y) = \int_{-\infty}^{\infty} \int_{-\infty}^y f(x,y) \, dx dy$$

❖ **INDEPENDENT RANDOM VARIABLES:**

✓ Two random variables X and Y are defined to be independent if

➤ $P(X = x, Y = y) = P_X(x) \cdot P_Y(y)$ If X and Y are discrete

➤ $f(x,y) = F_X(x) \cdot F_Y(y)$ If X and Y are continuous

✓ **Example:** The joint probability mass function (PMF) of X and Y is

	Y = 0	Y = 1	Y = 2
X = 0	0.1	0.04	0.02
X = 1	0.08	0.2	0.06
X = 2	0.06	0.14	0.3

✓ The marginal Probability Mass Function of X=0 is

$$P_X(X = 0) = 0.1 + 0.04 + 0.02 = 0.16$$

✓ The marginal Probability Mass Function of Y=0 is

$$P_Y(Y = 0) = 0.1 + 0.08 + 0.06 = 0.24$$

✓ $P_X(0) P_Y(0) = 0.16 \times 0.24 = 0.0384$ But $P(X = 0, Y = 0) = 0.1$

$$\therefore P(X = 0, Y = 0) \neq P_X(0) \cdot P_Y(0)$$

∴ X and Y are not independent random variables.

❖ **EXPECTED VALUE OF TWO DIMENSIONAL RANDOM VARIABLE:**

✓ Discrete case:

$$E(X) = \sum x_i P_X(x_i) \quad \& \quad E(Y) = \sum y_i P_Y(y_i)$$

✓ Continuous case:

$$E(X) = \iint_R x f(x,y) \, dx \, dy \quad \text{and} \quad E(Y) = \iint_R y f(x,y) \, dy \, dx \quad (\text{where R is given region})$$

METHOD – 6: EXAMPLES ON TWO DIMENSIONAL RANDOM VARIABLE

H	1	<p>X, Y are two random variables with joint mass function $P(x, y) = \frac{1}{27}(2x + y)$ where $x = 0, 1, 2$ and $y = 0, 1, 2$. Find the marginal probabilities.</p> <p>Answer: X: $\frac{3}{27}, \frac{9}{27}, \frac{15}{27}$ & Y: $\frac{6}{27}, \frac{9}{27}, \frac{12}{27}$</p>																												
C	2	<p>The joint probability mass function is given by $p(x, y) = k(2x + 3y)$, where $x = 0, 1, 2$ and $y = 1, 2, 3$. Find (a) k, (b) $P(x \leq 1, y \geq 2)$, (c) marginal probability, (d) expected value.</p> <p>Answer: (a) $k = \frac{1}{72}$, (b) 0.4722 (c) X: 0.25, 0.3333, 0.4167 & Y: 0.2083, 0.3333, 0.4583, (d) 1.1667</p>																												
H	3	<p>Let $P(X = 0, Y = 1) = 1/3, P(X = 1, Y = -1) = 1/3, P(X = 1, Y = 1) = \frac{1}{3}$. Is it the joint probability mass function of X and Y? If yes, find the marginal probability function of X and Y.</p> <p>Answer: yes, $P_X(0) = \frac{1}{3}, P_X(1) = \frac{2}{3}$ & $P_Y(-1) = \frac{1}{3}, P_Y(1) = \frac{2}{3}$</p>																												
H	4	<p>A two dimensional random variable (X, Y) have a bivariate distribution given by $P(X = x, Y = y) = \frac{x^2+y}{32}$ for $x = 0, 1, 2, 3$ & $y = 0, 1$. Find the marginal distributions of X and Y. Also, check the independence of X & Y.</p> <p>Answer: X: $\frac{1}{32}, \frac{3}{32}, \frac{9}{32}, \frac{19}{32}$ & Y: $\frac{14}{32}, \frac{18}{32}$. NO</p>																												
H	5	<p>For given joint probability distribution of X and Y, find $P(X \leq 1, Y = 2)$, $P(X \leq 1), P(Y \leq 3), P(X < 3, Y \leq 4)$. Also check the independence of X & Y.</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th></th> <th>Y = 1</th> <th>Y = 2</th> <th>Y = 3</th> <th>Y = 4</th> <th>Y = 5</th> <th>Y = 6</th> </tr> </thead> <tbody> <tr> <th>X = 0</th> <td>0</td> <td>0</td> <td>$\frac{1}{32}$</td> <td>$\frac{2}{32}$</td> <td>$\frac{2}{32}$</td> <td>$\frac{3}{32}$</td> </tr> <tr> <th>X = 1</th> <td>$\frac{1}{16}$</td> <td>$\frac{1}{16}$</td> <td>$\frac{1}{8}$</td> <td>$\frac{1}{8}$</td> <td>$\frac{1}{8}$</td> <td>$\frac{1}{8}$</td> </tr> <tr> <th>X = 2</th> <td>$\frac{1}{32}$</td> <td>$\frac{1}{32}$</td> <td>$\frac{1}{64}$</td> <td>$\frac{1}{64}$</td> <td>0</td> <td>$\frac{2}{64}$</td> </tr> </tbody> </table> <p>Answer: $\frac{1}{16}, \frac{7}{8}, \frac{23}{64}, \frac{9}{16}$. NO</p>		Y = 1	Y = 2	Y = 3	Y = 4	Y = 5	Y = 6	X = 0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	X = 1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	X = 2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$
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H	6	<p>The following table represents the joint probability distribution of discrete random variable (X, Y). Find $P(x \leq 2, Y = 3)$, $P(X + Y < 4)$ & $P(Y \leq 2)$.</p> <table border="1" data-bbox="252 315 1054 692"> <thead> <tr> <th></th> <th>Y = 1</th> <th>Y = 2</th> <th>Y = 3</th> <th>Y = 4</th> <th>Y = 5</th> <th>Y = 6</th> </tr> </thead> <tbody> <tr> <th>X = 0</th> <td>0</td> <td>0</td> <td>$\frac{1}{32}$</td> <td>$\frac{2}{32}$</td> <td>$\frac{2}{32}$</td> <td>$\frac{3}{32}$</td> </tr> <tr> <th>X = 1</th> <td>$\frac{1}{16}$</td> <td>$\frac{1}{16}$</td> <td>$\frac{1}{8}$</td> <td>$\frac{1}{8}$</td> <td>$\frac{1}{8}$</td> <td>$\frac{1}{8}$</td> </tr> <tr> <th>X = 2</th> <td>$\frac{1}{32}$</td> <td>$\frac{1}{32}$</td> <td>$\frac{1}{64}$</td> <td>$\frac{1}{64}$</td> <td>0</td> <td>$\frac{2}{64}$</td> </tr> </tbody> </table> <p>Answer: $\frac{11}{64}$, $\frac{3}{16}$, $\frac{13}{32}$</p>		Y = 1	Y = 2	Y = 3	Y = 4	Y = 5	Y = 6	X = 0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	X = 1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	X = 2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	
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C	7	<p>Suppose that 2 batteries are randomly chosen without replacement from the group of 12 batteries which contains 3 new batteries, 4 used batteries and 5 defective batteries. Let X denote the number of new batteries chosen and Y denote the number of used batteries chosen then, find the joint probability distribution.</p> <p>Answer: $P(0, 0) = 0.1515$, $P(1, 0) = 0.2273$, $P(2, 0) = 0.0455$, $P(0, 1) = 0.3030$, $P(1, 1) = 0.1818$, $P(0, 2) = 0.0909$</p>																													
C	8	<p>Check whether $f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y); & 0 \leq x < 2, 2 \leq y < 4 \\ 0 & ; \text{ otherwise} \end{cases}$ is probability density function or not ?</p> <p>Answer: yes</p>																													
H	9	<p>Let $f(x, y) = \begin{cases} Cxy & ; 0 < x < 4, 1 < y < 5 \\ 0 & ; \text{ otherwise} \end{cases}$ is the joint density function of two random variables X & Y, then find the value of C.</p> <p>Answer: $\frac{1}{96}$</p>																													
C	10	<p>For given P. d. f. $f(x, y) = \begin{cases} \frac{3}{4} + xy & ; 0 \leq x < 1, 0 \leq y < 1 \\ 0 & ; \text{ otherwise} \end{cases}$, find (a) joint Probability, (b) marginal Probability, (c) $P\left(0 \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}\right)$.</p> <p>Answer: (a) 1, (b) $X: \frac{3}{4} + \frac{X}{2}; (0 \leq x < 1)$ & $Y: \frac{3}{4} + \frac{Y}{2}; (0 \leq y < 1)$, (c) $\frac{13}{64}$</p>																													

H	11	<p>Suppose, two dimensional continuous random variable (X, Y) has PDF given by $f(x,y) = \begin{cases} 6x^2y & ; 0 < x < 1, 0 < y < 1 \\ 0 & ; \text{elsewhere} \end{cases}$</p> <p>(a) Verify $\int_0^1 \int_0^1 f(x,y) dx dy = 1$.</p> <p>(b) Find $P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right)$ & $P(X + Y < 1)$.</p> <p>Answer: $\frac{3}{8}, \frac{1}{10}$</p>																	
C	12	<p>The joint pdf of a two-dimensional random variable (X, Y) is given by $f(x,y) = \begin{cases} 2 & ; 0 < x < 1, 0 < y < x \\ 0 & ; \text{elsewhere} \end{cases}$</p> <p>Find the marginal density function of X and Y.</p> <p>Answer: $f_X(x) = 2x ; 0 < x < 1$ & $f_Y(y) = 2(1 - y) ; 0 < y < 1$</p>																	
C	13	<p>Check the independence of X and Y for the following PDF.</p> $f(x,y) = \begin{cases} \frac{1}{4}(1 + xy) & ; -1 < x < 1, -1 < y < 1 \\ 0 & ; \text{elsewhere} \end{cases}$ <p>Answer: NO</p>																	
C	14	<p>The random variables X and Y have the following joint probability distribution. What is the expected value of X and Y?</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td></td> <td>Y = 0</td> <td>Y = 1</td> <td>Y = 2</td> </tr> <tr> <td>X = 0</td> <td>0.2</td> <td>0.1</td> <td>0.2</td> </tr> <tr> <td>X = 1</td> <td>0</td> <td>0.2</td> <td>0.1</td> </tr> <tr> <td>X = 2</td> <td>0.1</td> <td>0</td> <td>0.1</td> </tr> </table> <p>Answer: 0.7, 1.1</p>		Y = 0	Y = 1	Y = 2	X = 0	0.2	0.1	0.2	X = 1	0	0.2	0.1	X = 2	0.1	0	0.1	
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C	15	<p>Consider the joint probability density function for X and Y to be $f(x,y) = x^2 y^3 ; 0 < x < 1$ & $0 < y < x$, find the expected value of X.</p> <p>Answer: $\frac{1}{32}$</p>																	

