HEAT TRANSFER FROM EXTENDED SURFACES

Course Contents

3.1 Introduction
3.2 Steady flow of heat along a rod (governing differential equation)
3.3 Heat dissipation from an infinitely long fin
3.4 Heat dissipation from a fin insulated at the tip
3.5 Heat dissipation from a fin losing heat at the tip
3.6 Fin performance
3.7 Thermometric well
3.8 Solved Numerical
3.9 References
3. Heat Transfer from Extended Surface

3.1 Introduction

- Heat transfer between a solid surface and a moving fluid is governed by the Newton’s cooling law: \( Q_{\text{conv}} = h A_s (T_0 - T_a) \), where \( T_0 \) is the surface temperature and \( T_a \) is the fluid temperature.
- Therefore, to increase the convective heat transfer, one can
  i. Increase the temperature difference \( (T_0 - T_a) \) between the surface and the fluid.
  ii. Increase the convection coefficient \( h \). This can be accomplished by increasing the fluid flow over the surface since \( h \) is a function of the flow velocity and the higher the velocity, the higher the \( h \).
  iii. Increase the contact surface area \( A_s \)
- Many times, when the first option is not in our control and the second option (i.e. increasing \( h \)) is already stretched to its limit, we are left with the only alternative of increasing the effective surface area by using fins or extended surfaces.
- Fins are protrusions from the base surface into the cooling fluid, so that the extra surface of the protrusions is also in contact with the fluid.
- Most of you have encountered cooling fins on air-cooled engines (motorcycles, portable generators, etc.), electronic equipment (CPUs), automobile radiators, air conditioning equipment (condensers) and elsewhere

3.2 Steady Flow of Heat Along A Rod (Governing Differential Equation)

- Consider a straight rectangular or pin fin protruding from a wall surface (figure 3.1a and figure 3.1b).
- The characteristic dimensions of the fin are its length \( L \), constant cross-sectional area \( A_c \) and the circumferential parameter \( P \).
Thus for a rectangular fin

\[ A_c = b\delta \; ; \; P = 2(b + \delta) \]

And for a pin fin

\[ A_c = \frac{\pi}{4} d^2 \; ; \; P = \pi d \]

The temperature at the base of the fin is \( T_0 \) and the temperature of the ambient fluid into which the rod extends is considered to be constant at temperature \( T_a \).

The base temperature \( T_0 \) is highest and the temperature along the fin length goes on diminishing.

Analysis of heat flow from the finned surface is made with the following assumptions:

i. Thickness of the fin is small compared with the length and width; temperature gradients over the cross-section are neglected and heat conduction treated one dimensional

ii. Homogeneous and isotropic fin material; the thermal conductivity \( k \) of the fin material is constant

iii. Uniform heat transfer coefficient \( h \) over the entire fin surface

iv. No heat generation within the fin itself

v. Joint between the fin and the heated wall offers no bond resistance; temperature at root or base of the fin is uniform and equal to temperature \( T_0 \) of the wall

vi. Negligible radiation exchange with the surroundings; radiation effects, if any, are considered as included in the convection coefficient \( h \)
vii Steady state heat dissipation

- Heat from the heated wall is conducted through the fin and convected from the sides of the fin to the surroundings.
- Consider infinitesimal element of the fin of thickness dx at a distance x from base wall as shown in figure 3.2.

\[ Q_x = -k A_c \left( \frac{dt}{dx} \right)_x \]  \hspace{1cm} (3.1)

\[ Q_{x+dx} = -k A_c \left( \frac{dt}{dx} \right)_{x+dx} \]

\[ = -k A_c \frac{d}{dx} \left( t + \frac{dt}{dx} dx \right) \]  \hspace{1cm} (3.2)

- Heat convected out of the element between the planes x and (x + dx)

\[ Q_{conv} = h (P dx)(t - t_a) \]  \hspace{1cm} (3.3)

- Here temperature t of the fin has been assumed to be uniform and non-variant for the infinitesimal element.
- According to first law of thermodynamic, for the steady state condition, heat transfer into element is equal to heat transfer from the element

\[ Q_x = Q_{x+dx} + Q_{conv} \]
Heat Transfer from Extended Surface

\[-k A_c \frac{dt}{dx} = -k A_c \frac{d}{dx} \left( t + \frac{dt}{dx} \right) + h (P \frac{dx}{dt})(t - t_a)\]

\[-k A_c \frac{dt}{dx} = -k A_c \frac{dt}{dx} - k A_c \frac{d^2 t}{dx^2} dx + h (P \frac{dx}{dt})(t - t_a)\]

- Upon arrangement and simplification

\[\frac{d^2 t}{dx^2} - \frac{hP}{k A_c} (t - t_a) = 0 \quad \text{(3.4)}\]

Let, \( \theta(x) = t(x) - t_a \)

- As the ambient temperature is constant, so differentiation of the equation is

\[\frac{d\theta}{dx} = \frac{dt}{dx}; \quad \frac{d^2 \theta}{dx^2} = \frac{d^2 t}{dx^2}\]

Thus

\[\frac{d^2 \theta}{dx^2} - m^2 \theta = 0 \quad \text{(3.5)}\]

Where

\[m = \sqrt{\frac{hP}{k A_c}}\]

- Equations 3.4 and 3.5 provide a general form of the energy equation for one dimensional heat dissipation from an extended surface.

- The general solution of this linear homogeneous second order differential equation is of the form

\[\theta = C_1 e^{mx} + C_2 e^{-mx} \quad \text{(3.6)}\]

- The constant \( C_1 \) and \( C_2 \) are to be determined with the aid of relevant boundary conditions. We will treat the following four cases:
  i. Heat dissipation from an infinitely long fin
  ii. Heat dissipation from a fin insulated at the tip
  iii. Heat dissipation from a fin losing heat at the tip

### 3.3 Heat Dissipation From an Infinitely Long Fin

- Governing differential equation for the temperature distribution along the length of the fin is given as,

\[\theta = C_1 e^{mx} + C_2 e^{-mx} \quad \text{(3.7)}\]

- The relevant boundary conditions are
Fig. 3.3 Temperature distribution along the infinite long fin

- Temperature at the base of fin equals the temperature of the surface to which the fin is attached.
  \[ t = t_0 \text{ at } x = 0 \]

- In terms of excess temperature
  \[ t - t_a = t_0 - t_a \text{ at } x = 0 \]
  or
  \[ \theta = \theta_0 \text{ at } x = 0 \]

- Substitution of this boundary condition in equation gives:
  \[ C_1 + C_2 = \theta_0 \]
  \[ \theta = \theta_0 \text{ at } x = 0 \]

- Temperature at the end of an infinitely long fin equals that of the surroundings.
  \[ t = t_a \text{ at } x = \infty \]
  \[ \theta = 0 \text{ at } x = \infty \]

- Substitution of this boundary condition in equation gives:
  \[ C_1 e^{m\infty} + C_2 e^{-m\infty} = 0 \]

  Since the term \( C_2 e^{-m\infty} \) is zero, the equality is valid only if \( C_1 = 0 \). Then from equation 3.8 \( C_2 = \theta_0 \).

- Substituting these values of constant \( C_1 \) and \( C_2 \) in equation 3.7, following expression is obtained for temperature distribution along the length of the fin.
  \[ \theta = \theta_0 e^{-m\infty}; \quad (t - t_a) = (t_0 - t_a)e^{-m\infty} \]
  \[ \text{(3.10)} \]

- **Heat transfer from fin**

- Heat transfer to the fin at base of the fin must equal to the heat transfer from the surface of the fin by convection. Heat transfer to the fin at base is given as
  \[ Q_{fin} = -k A_c \left( \frac{dt}{dx} \right)_{x=0} \]
  \[ \text{(3.11)} \]

- From the expression for the temperature distribution (Equation 3.10)
  \[ t = t_\infty + (t_0 - t_a)e^{-m\infty} \]
\[
\left( \frac{dt}{dx} \right)_{x=0} = [-m(t_0 - t_a)e^{-mx}]_{x=0} = -m(t_0 - t_a)
\]

- Substitute the value of \( \left( \frac{dt}{dx} \right)_{x=0} \) in the equation 3.11
  \[
  \therefore Q_{fin} = kA_c m(t_0 - t_a)
  \]
  But
  \[
  m = \sqrt{\frac{hP}{kA_c}}
  \]
  \[
  \therefore Q_{fin} = \sqrt{PhkA_c}(t_0 - t_a) - - - - - - - (3.12)
  \]

- The temperature distribution (Equation 3.10) would suggest that the temperature drops towards the tip of the fin.
- Hence area near the fin tip is not utilized to the extent as the lateral area near the base. Obviously an increase in length beyond certain point has little effect on heat transfer.
- So it is better to use tapered fin as it has more lateral area near the base where the difference in temperature is high.

- **Ingen-Hausz Experiment**

  - Heat flow rates through solids can be compared by having an arrangement consisting essentially of a box to which rods of different materials are attached (Ingen-Hausz experiment).
  - The rods are of same length and area of cross-section (same size and shape); their outer surfaces are electroplated with the same material and are equally polished.
  - This is to ensure that for each rod, the surface heat transfer will be same. Heat flow from the box along the rod would melt the wax for a distance which would depend upon the rod material. Let
    \[
    \theta_0 = \text{excess of temperature of the hot bath above the ambient temperature}
    \]
    \[
    \theta_m = \text{excess of temperature of melting point of wax above the ambient temperature}
    \]
    \[
    l_1, l_2, l_3 \ldots \ldots \ldots \ldots \text{ = lengths upto which wax melts.}
    \]
  - Then for different rods (treating each as fin of infinite length),
    \[
    \theta_m = \theta_0 e^{-m_1 l_1}
    \]
3. Heat Transfer from Extended Surface

\[ m_1 l_1 = m_2 l_2 = m_3 l_3 \]

So

\[ \frac{l_1}{\sqrt{k_1}} = \frac{l_2}{\sqrt{k_2}} = \frac{l_3}{\sqrt{k_3}} = \text{constant} \quad (3.13) \]

Thus, the thermal conductivity of the material of the rod is directly proportional to the square of the length up to which the wax melts on the rod.

3.4 Heat Dissipation From a Fin Insulated At The Tip

- The fin is of any finite length with the end insulated and so no heat is transferred from the tip.
- Therefore, the relevant boundary conditions are:
  - Temperature at the base of fin equals the temperature of the surface to which the fin is attached.
  - In terms of excess temperature

\[ t = t_0 \text{ at } x = 0 \]

\[ \frac{k_1}{l_1^2} = \frac{k_2}{l_2^2} = \frac{k_3}{l_3^2} = \text{constant} \]

Fig. 3.5 Heat dissipation from a fin insulated at the tip
\[ t - t_a = t_0 - t_a \]
\[ \theta = \theta_0 \text{ at } x = 0 \]

- Substitution of this boundary condition in equation 3.6 gives:
  \[ C_1 + C_2 = \theta_0 - \theta - \theta \text{ at } x = 0 \]

- As the tip of fin is insulated, temperature gradient is zero at end of the fin.
  \[ \frac{dt}{dx} = 0 \text{ at } x = L \]

But

\[ t - t_a = C_1 e^{mx} + C_2 e^{-mx} \]

\[ \frac{dt}{dx} = mC_1 e^{mx} - mC_2 e^{-mx} \]

\[ \left( \frac{dt}{dx} \right)_{x=L} = mC_1 e^{mL} - mC_2 e^{-mL} = 0 \]

\[ \therefore C_1 e^{mL} - C_2 e^{-mL} = 0 \]

- Substitute the value of \( C_1 \) from equation 3.14 into equation 3.15

\[ (\theta_0 - C_2)e^{mL} - C_2 e^{-mL} = 0 \]

\[ \theta_0 e^{mL} - C_2 e^{mL} - C_2 e^{-mL} = 0 \]

\[ \theta_0 e^{mL} - C_2(e^{mL} + e^{-mL}) = 0 \]

\[ \theta_0 e^{mL} = C_2(e^{mL} + e^{-mL}) \]

\[ C_2 = \theta_0 \frac{e^{mL}}{e^{mL} + e^{-mL}} \]

- Substitute the value of \( C_2 \) in equation 3.14, we get

\[ C_1 = \theta_0 \frac{e^{-mL}}{e^{mL} + e^{-mL}} \]

- Substitute the values of constant in equation 3.6, expression for temperature distribution along the length of the fin is obtained

\[ \frac{\theta}{\theta_0} = \frac{e^{m(L-x)} + e^{-m(L-x)}}{e^{mL} + e^{-mL}} \]

- In terms of hyperbolic function, expression is given as

\[ \frac{\theta}{\theta_0} = \frac{t - t_a}{t_0 - t_a} = \frac{\cosh m(L - x)}{\cosh mL} \]

- The rate of heat flow from the fin is equal to the heat conducted to the fin at the base, so heat flow from the fin is given by

\[ Q_{\text{fin}} = -k A_c \left( \frac{dt}{dx} \right)_{x=0} \]

- From the expression for the temperature distribution (Equation 3.18)

\[ t - t_a = (t_0 - t_a) \frac{\cosh m(L - x)}{\cosh mL} \]
\[ \frac{dt}{dx} = (t_0 - t_a) \frac{\sinh m(L - x)}{\cosh(mL)}(-m) \]

\[ \left( \frac{dt}{dx} \right)_{x=0} = -m(t_0 - t_a) \tanh(mL) \] (3.20)

Substitute the value of equation 3.20 in equation 3.19, we get

\[ Q_{\text{fin}} = k A_c m(t_0 - t_a) \tanh(mL) \]

But

\[ m = \sqrt{\frac{hP}{k A_c}} \]

\[ \therefore Q_{\text{fin}} = \sqrt{Phk A_c} (t_0 - t_a) \tanh(mL) \] (3.21)

### 3.5 Heat Dissipation From a Fin Losing Heat At The Tip

- The fin tips, in practice, are exposed to the surroundings. So heat may be transferred by convection from the fin tip.

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![Diagram showing heat dissipation from a fin losing heat at the tip](image)

**Fig. 3.6 Heat dissipation from fin losing heat at the tip**

- Therefore, relevant boundary conditions are
- Temperature at the base of fin equals the temperature of the surface to which the fin is attached.

\[ t = t_0 \text{ at } x = 0 \]

- In terms of excess temperature

\[ t - t_a = t_0 - t_a \]

or

\[ \theta = \theta_0 \text{ at } x = 0 \]

- Substitution of this boundary condition in equation 3.6 gives:

\[ C_1 + C_2 = \theta_0 \] (3.22)
As the fin is losing heat at the tip, i.e., the heat conducted to the fin at \( x = L \) equals the heat convected from the end to the surroundings

\[-k A_c \left( \frac{dt}{dx} \right)_{x=L} = h A_s (t - t_a)\]

At the tip of fin, the cross sectional area for heat conduction \( A_c \) equals the surface area \( A_s \) from which the convective heat transport occurs. Thus

\[\frac{dt}{dx} = -\frac{h\theta}{k} \text{ at } x = L \tag{3.23}\]

Governing differential equation of fin is given as

\[t - t_a = C_1 e^{mx} + C_2 e^{-mx}\]

\[\therefore \frac{dt}{dx} = mC_1 e^{mx} - mC_2 e^{-mx}\]

\[\left( \frac{dt}{dx} \right)_{x=L} = mC_1 e^{mL} - mC_2 e^{-mL}\]

Substitute above value in equation 3.23, we get

\[mC_1 e^{mL} - mC_2 e^{-mL} = -\frac{h\theta}{k}\]

\[C_1 e^{mL} - C_2 e^{-mL} = -\frac{h\theta}{km}\tag{3.24}\]

But, \( \theta = C_1 e^{mL} + C_2 e^{-mL} \) at \( x = L \)

Substitute this value in equation 3.24

\[C_1 e^{mL} - C_2 e^{-mL} = -\frac{h}{km} [C_1 e^{mL} + C_2 e^{-mL}]\]

Substitute the value of \( C_2 \) from equation 3.22 in above equation

\[C_1 e^{mL} - (\theta_0 - C_1) e^{-mL} = -\frac{h}{km} [C_1 e^{mL} + (\theta_0 - C_1) e^{-mL}]\]

\[C_1 e^{mL} - \theta_0 e^{-mL} + C_1 e^{-mL} = -\frac{h}{km} [C_1 e^{mL} + \theta_0 e^{-mL} - C_1 e^{-mL}]\]

\[C_1 e^{mL} - \theta_0 e^{-mL} + C_1 e^{-mL} = -\frac{h}{km} C_1 e^{mL} - \frac{h}{km} \theta_0 e^{-mL} + \frac{h}{km} C_1 e^{-mL}\]

\[C_1 e^{mL} + C_1 e^{-mL} + \frac{h}{km} C_1 e^{mL} - \frac{h}{km} C_1 e^{-mL} = \theta_0 e^{-mL} - \frac{h}{km} \theta_0 e^{-mL}\]

\[C_1 \left[e^{mL} + e^{-mL} + \frac{h}{km} (e^{mL} - e^{-mL})\right] = \theta_0 e^{-mL} \left[1 - \frac{h}{km}\right]\]

\[\therefore C_1 = \frac{\theta_0 e^{-mL} \left[1 - \frac{h}{km}\right]}{(e^{mL} + e^{-mL}) + \frac{h}{km} (e^{mL} - e^{-mL})}\]

And
3. Heat Transfer from Extended Surface

\[ C_2 = \theta_s - C_1 = \theta_0 - \frac{\theta_0 e^{-ml} \left[ 1 - \frac{h}{km} \right]}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \]

\[ = \theta_0 \left[ 1 - \frac{e^{-ml} \left( 1 - \frac{h}{km} \right)}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \right] \]

\[ = \theta_0 \left[ \frac{(e^{ml} + e^{-ml}) + \frac{h}{km} e^{ml} - \frac{h}{km} e^{-ml} - e^{-ml} \left( 1 - \frac{h}{km} \right)}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \right] \]

\[ = \theta_0 \left[ \frac{e^{ml} + \frac{h}{km} e^{ml}}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \right] \]

\[ C_2 = \frac{\theta_0 \left( 1 + \frac{h}{km} \right) e^{ml}}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \quad (3.25) \]

Substituting these values of constants \( C_1 \) and \( C_2 \) in equation 3.6, one obtains the following expression for temperature distribution along the length of the fin.

\[ \theta = \frac{\theta_0 e^{-ml} \left[ 1 - \frac{h}{km} \right]}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} e^{mx} \]

\[ + \frac{\theta_0 \left( 1 + \frac{h}{km} \right) e^{ml}}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} e^{-mx} \]

\[ \theta \frac{e^{m(l-x)}}{\theta_0} = \frac{e^{-m(l-x)} \left( 1 - \frac{h}{km} \right) + \left( 1 + \frac{h}{km} \right) e^{m(l-x)}}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \]

\[ \theta \frac{e^{-m(l-x)} - \frac{h}{km} e^{-m(l-x)} + e^{m(l-x)} + \frac{h}{km} e^{m(l-x)}}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \]

\[ \theta \frac{e^{m(l-x)} + e^{-m(l-x)} + \frac{h}{km} (e^{ml} - e^{-ml})}{(e^{ml} + e^{-ml}) + \frac{h}{km} (e^{ml} - e^{-ml})} \]

Expressing in terms of hyperbolic functions

\[ \theta = \frac{t - t_a}{t_0 - t_a} \quad \frac{\cosh m(L-x) + \frac{h}{km} \sinh m(L-x)}{\cosh(mL) + \frac{h}{km} \sinh(mL)} \quad (3.26) \]

The rate of heat flow from the fin is equal to the heat conducted to the fin at the base, so heat flow from the fin is given by
\[ Q_{\text{fin}} = -k \ A_c \left( \frac{dt}{dx} \right)_{x=0} \]  

- From the expression for the temperature distribution (Equation 3.26)

\[
t - t_a = (t_0 - t_a) \left[ \frac{\cosh m(L - x) + \frac{h}{km} \sinh m(L - x)}{\cosh (mL) + \frac{h}{km} \sinh (mL)} \right] 
\]

\[
\frac{dt}{dx} = (t_0 - t_a) \left[ \frac{-m \sinh (mL - x) - m \left( \frac{h}{km} \cosh (mL - x) \right)}{\cosh (mL) + \frac{h}{km} \sinh (mL)} \right] 
\]

\[
\left( \frac{dt}{dx} \right)_{x=0} = -m(t_0 - t_a) \left[ \frac{\sinh (mL) + \frac{h}{km} \cosh (mL)}{\cosh (mL) + \frac{h}{km} \sinh (mL)} \right] 
\]

- Substitute this value in equation 3.27

\[ Q_{\text{fin}} = k A_c m(t_0 - t_a) \left[ \frac{\sinh (mL) + \frac{h}{km} \cosh (mL)}{\cosh (mL) + \frac{h}{km} \sinh (mL)} \right] \]

But,

\[ m = \sqrt{\frac{hP}{k A_c}} \]

\[ = \sqrt{Phk A_c(t_0 - t_a)} \left[ \frac{\sinh (mL) + \frac{h}{km} \cosh (mL)}{\cosh (mL) + \frac{h}{km} \sinh (mL)} \right] \]

\[ = \sqrt{Phk A_c(t_0 - t_a)} \left[ \frac{\tanh (mL) + \frac{h}{km}}{1 + \frac{h}{km} \tanh (mL)} \right] \]  

### 3.6 Fin Performance

- It is necessary to evaluate the performance of fins to achieve minimum weight or maximum heat flow etc.

- Fin effectiveness and fin efficiency are some methods used for performance evaluation of fins

- **Efficiency of fin:**

  - It relates the performance of an actual fin to that of an ideal or fully effective fin.

  - In reality, temperature of fin drop along the length of fin, and thus the heat transfer from the fin will be less because of the decreasing temperature difference towards the tip of fin.

  - A fin will be most effective, i.e., it would dissipate heat at maximum rate if the entire fin surface area is maintained at the base temperature as shown in figure 3.7
3. Heat Transfer from Extended Surface

Heat Transfer

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Fig. 3.7 Ideal and actual temperature distribution in a fin

\[ \eta_f = \frac{\text{actual heat transfer rate from fin}}{\text{maximum possible heat transfer rate from fin}} \]

- Thus for a fin insulated at tip

\[ \eta_f = \frac{\sqrt{P \cdot h \cdot k \cdot A_c \cdot (t_0 - t_a) \cdot \tanh(mL)}}{h(P) \cdot (t_0 - t_a)} \]

- The parameter PL represents the total surface area exposed for convective heat flow. Upon simplification,

\[ \eta_f = \frac{\tanh(mL)}{\sqrt{P} \cdot h \cdot k \cdot A_c \cdot L} = \frac{\tanh(mL)}{mL} \]

- Following points are noted down from the above equation

  i. For a very long fin

\[ \frac{\tanh(mL)}{mL} \rightarrow 1 \text{ large number} \]

  - Obviously the fin efficiency drops with an increase in its length.

  - For small values of \( mL \), the fin efficiency increases. When the length is reduced to zero, then,

\[ \frac{\tanh(mL)}{mL} \rightarrow \frac{mL}{mL} = 1 \]

  - Thus the fin efficiency reaches its maximum value of 100% for a trivial value of \( L = 0 \), i.e., no fin at all.

  - Actually efficiency of fin is used for the design of the fin but it is used for comparison of the relative merits of fin of different geometries or material.

  - Note that fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles, and thus are more suitable for applications requiring minimum weight such as space applications.

  - An important consideration in the design of finned surfaces is the selection of the proper fin length \( L \).

  - Normally the longer the fin, the larger the heat transfer area and thus the higher the rate of heat transfer from the fin.

  - But also the larger the fin, the bigger the mass, the higher the price, and the larger the fluid friction.
Therefore, increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.

Also, the fin efficiency decreases with increasing fin length because of the decrease in fin temperature with length.

Fin lengths that cause the fin efficiency to drop below 60 percent usually cannot be justified economically and should be avoided.

The efficiency of most fins used in practice is above 90 percent.

**Effectiveness of fin ($\varepsilon_f$):**

Fins are used to increase the heat transfer. And use of fin can not be recommended unless the increase in heat transfer justifies the added cost of fin.

In fact, use of fin may not ensure the increase in heat transfer. Effectiveness of fin gives the increase in heat transfer with fin relative to no fin case.

It represents the ratio of the fin heat transfer rate to the heat transfer rate that would exist without a fin.

$$\varepsilon_f = \frac{\text{heat transfer with fin}}{\text{heat transfer without fin}}$$

Figure 3.8 shows the base heat transfer surface before and after the fin has been attached.

Heat transfer through the root area $A_c$ before the fin attached is:

$$Q = hA_c(t_0 - t_a)$$

After the attachment of an infinitely long fin, the heat transfer rate through the root area becomes:

$$Q_{fin} = \sqrt{Phk}A_c(t_0 - t_a)$$

So, effectiveness of fin is given as

$$\therefore \varepsilon_f = \frac{\sqrt{Phk}A_c(t_0 - t_a)}{hA_c(t_0 - t_a)}$$
\[ \varepsilon_f = \sqrt{\frac{P_k}{hA_c}} \]  

- Following conclusions are given from the effectiveness of the fin
  
  i. If the fin is used to improve heat dissipation from the surface, then the fin effectiveness must be greater than unity. That is,
  
  \[ \sqrt{\frac{P_k}{hA_c}} > 1 \]

  But literature suggests that use of fins on surface is justified only if the ratio \( P_k/hA_c \) is greater than 5.

  ii. To improve effectiveness of fin, fin should be made from high conductive material such as copper and aluminium alloys. Although copper is superior to aluminium regarding to the thermal conductivity, yet fins are generally made of aluminium because of their additional advantage related to lower cost and weight.

  iii. Effectiveness of fin can also be increased by increasing the ratio of perimeter to the cross sectional area. So it is better to use more thin fins of closer pitch than fewer thicker fins at longer pitch.

  iv. A high value of film coefficient has an adverse effect on effectiveness. So fins are used with the media with low film coefficient. Therefore, in liquid – gas heat exchanger, such as car radiator, fins are placed on gas side.

- Relation between efficiency of fin and effectiveness of fin
  
  \[ \eta_f = \frac{\sqrt{PhkA_c(t_0 - t_a)\tanh(mL)}}{h(PL)(t_0 - t_a)} \]

  \[ \varepsilon_f = \frac{\sqrt{PhkA_c(t_0 - t_a)\tanh(mL)}}{hA_c(t_0 - t_a)} \]

  \[ \therefore \varepsilon_f = \eta_f \frac{h(PL)(t_0 - t_a)}{hA_c(t_0 - t_a)} = \eta_f \frac{(PL)}{A_c} \]

  \[ \therefore \varepsilon_f = \eta_f \frac{\text{surface area of fin}}{\text{cross-sectional area of fin}} \]

- An increase in the fin effectiveness can be obtained by extending the length of fin but that rapidly becomes a losing proposition in term of efficiency.

### 3.7 Thermometric Well

- Figure 3.9 shows an arrangement which is used to measure the temperature of gas flowing through a pipeline.
- A small tube called thermometric well is welded radially into the pipeline. The well is partially filled with some liquid and the thermometer is immersed into this liquid.
- When the temperature of gas flowing through the pipe is higher than the ambient temperature, the heat flows from the hot gases towards the tube walls along the well. This may cause temperature at the bottom of well to become colder than the gas flowing around.
- So the temperature indicated by the thermometer will not be the true temperature of the gas.
- The error in the temperature measurement is estimated with the help of the theory of extended surfaces.

![Diagram of Thermometric Well]

**Fig. 3.9 Thermometric well**

- The thermometric well can be considered as a hollow fin with insulated tip. Temperature distribution is obtained as

\[
\theta_x = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(l - x)}{\cosh(mL)}
\]

- Where \(t_0\) is the temperature of pipe wall, \(t_a\) is the temperature of hot gas or air flowing through the pipeline, and \(t_x\) is the temperature at any distance \(x\) measured from pipe wall along the thermometric well.

- If \(x = l\) then

\[
\frac{t_l - t_a}{t_0 - t_a} = \frac{\cosh m(l - l)}{\cosh(mL)} = \frac{1}{\cosh(mL)}
\]

- Where \(t_l\) is the temperature recorded by the thermometer at the bottom of the well.
The perimeter of the protective well \( P = \pi (d + 2\delta) \approx \pi d \), and its cross-sectional area \( A_c = \pi d \delta \). Therefore
\[
\frac{P}{A_c} = \frac{\pi d}{\pi d \delta} = \frac{1}{\delta}
\]

Then
\[
m = \sqrt{\frac{h P}{k A_c}} = \sqrt{\frac{h}{k \delta}} (3.33)
\]

From the equation 3.33 it is clear that diameter of the well does not have any effect on temperature measurement by the thermometer.

The error can be minimized by
i. Lagging the tube so that conduction of heat along its length is arrested.
ii. Increasing the value of parameter \( ml \)

For a rectangle fin \( m = \sqrt{2h/k\delta} \).

An increasing in \( m \) can be affected by using a thinner tube of low thermal conductivity or by increasing the convection co-efficient through finning the manometric well.

The operative length \( l \) is increased by inking the pocket and setting its projection beyond the pipe axis.

### 3.8 Solved Numerical

Ex. 3.1.

A cooper rod 0.5 cm diameter and 50 cm long protrudes from a wall maintained at a temperature of 500°C. The surrounding temperature is 30°C. Convective heat transfer coefficient is 40 W/m²K and thermal conductivity of fin material is 300 W/mK. Show that this fin can be considered as infinitely long fin. Determine total heat transfer rate from the rod.

Solution:

*Given data:*
\[ d = 0.5 \text{ cm} = 0.5 \times 10^{-2} \text{ m}, \quad L = 50 \text{ cm} = 0.5 \text{ m}, \quad t_0 = 500^\circ\text{C}, \quad t_a = 30^\circ\text{C}, \quad h = 40 \text{ W/m}^2\text{K}, \quad k = 300 \text{ W/mK} \]

\[ A_c = \frac{\pi}{4} d^2 = \frac{\pi}{4} (0.5 \times 10^{-2})^2 = 1.96 \times 10^{-5} \text{ m}^2 \]

\[ p = \frac{\pi d}{A_c} = \frac{4}{d} \]

\[ m = \sqrt{\frac{hp}{kA_c}} = \sqrt{\frac{4h}{300 \times 0.5 \times 10^{-2}}} = 10.32 \text{ m}^{-1} \]

Fin can be considered as infinite long fin, if heat loss from the infinitely long rod is equal to heat loss from insulated tip rod.

Heat loss from infinitely long rod is given by

\[ Q_{fin} = k A_c m(t_0 - t_a) \]

and heat loss from the insulated tip fin is given by

\[ Q_{fin} = k A_c m(t_0 - t_a) \tanh(mL) \]

These expressions provide equivalent results if \( \tanh(mL) \geq 0.99 \) or \( mL \geq 2.65 \)

Hence the rod can be considered infinite if

\[ L \geq \frac{2.65}{m} \geq \frac{2.65}{10.32} \geq 0.256 \text{ m} \]

Since length of the rod (0.5 m) is greater than 0.256 m, rod can be considered as infinitely long rod.

Heat loss from infinitely long rod is given by

\[ Q_{fin} = 300 \times 1.96 \times 10^{-5} \times 10.32 \times (500 - 30) = 28.57 \text{ W} \]

**Ex. 3.2.**

Two rods A and B of equal diameter and equal length, but of different materials are used as fins. The both rods are attached to a plain wall maintained at 160\(^\circ\text{C}\), while they are exposed to air at 30\(^\circ\text{C}\). The end temperature of rod A is 100\(^\circ\text{C}\) while that of the rod B is 80\(^\circ\text{C}\). If thermal conductivity of rod A is 380 W/m-K, calculate the thermal conductivity of rod B. These fins can be assumed as short with end insulated.

**Solution:**

**Given data:**

Both rods are similar in their shape and size, connected to same wall and exposed to same environment. So, for both the rods area and perimeters are equal and following parameters are same.

\[ t_0 = 180^\circ\text{C}, \quad t_a = 30^\circ\text{C}, \quad h_A = h_B \]

For rod A: \( t_{LA} = 100^\circ\text{C}, \quad k_A = 380 \text{ W/mK} \)

For rod B: \( t_{LB} = 80^\circ\text{C}, \quad k_B = ? \)

Temperature distribution for insulated tip fin is given by

\[ \frac{t - t_a}{t_0 - t_a} = \frac{\cosh mL - x}{\cosh(mL)} \]

And temperature at the free end, \( x = L \)
3. Heat Transfer from Extended Surface

\[ \frac{t_L - t_a}{t_0 - t_a} = \frac{1}{\cosh(mL)} \]

For rod A

\[
\begin{align*}
100 - 30 & = \frac{1}{160 - 30} \\
160 - 30 & = 130
\end{align*}
\]

\[ \therefore \cosh(m_A L) = \cosh^{-1}(\frac{130}{70}) = 1.857 \]

\[ \therefore m_A L = \cosh^{-1}(1.857) = 1.23 \]

For rod B

\[
\begin{align*}
80 - 30 & = \frac{1}{160 - 30} \\
160 - 30 & = 130
\end{align*}
\]

\[ \therefore \cosh(m_B L) = \cosh^{-1}(\frac{130}{50}) = 2.6 \]

\[ \therefore m_B L = \cosh^{-1}(2.6) = 1.609 \]

From above two calculation

\[ \frac{m_A L}{m_B L} = \frac{1.23}{1.609} = 0.764 \]

\[ \sqrt{\frac{p h_A / k_A A_c}{p h_B / k_B A_c}} = 0.764 \]

\[ \sqrt{\frac{380}{k_B}} = 0.764 \]

\[ \therefore k_B = (0.764)^2 \times 380 = 221.94 \text{ W/mK} \]

Ex. 3.3.

A steel rod (k=30 W/m°C), 12 mm in diameter and 60 mm long, with an insulated end is to be used as spine. It is exposed to surrounding with a temperature of 60°C and heat transfer coefficient of 55 W/m²°C. The temperature at the base is 100°C. Determine: (i) The fin effectiveness (ii) The fin efficiency (iii) The temperature at the edge of the spine (iv) The heat dissipation.

**Solution:**

Given data:

\[ d = 12 \text{ mm} = 12 \times 10^{-3} \text{ m}, \quad L = 60 \text{ mm} = 0.06 \text{ m}, \quad t_0 = 100°C, \quad t_a = 60°C, \quad h = 55 \text{ W/m}^2\text{K}, \quad k = 30 \text{ W/mK} \]

\[ \frac{p}{A_c} = \frac{\pi d}{4 d^2} = \frac{4}{d} \]

\[ m = \sqrt{\frac{h p}{k A_c}} = \sqrt{\frac{4h}{kd}} = \sqrt{\frac{4 \times 55}{30 \times 12 \times 10^{-3}}} = 24.72 \text{ m}^{-1} \]

\[ A_c = \frac{\pi}{4} d^2 = \frac{\pi}{4} (12 \times 10^{-3})^2 = 1.13 \times 10^{-4} \text{m}^2 \]

i. Effectiveness of the fin
\[ \epsilon_f = \frac{\text{heat transfer with fin}}{\text{heat transfer without fin}} \]
\[ \epsilon_f = \frac{\sqrt{P h k A_c} (t_0 - t_a) \tanh(mL)}{h A_c (t_0 - t_a)} \]
\[ \epsilon_f = \frac{p k}{h A_b} \tanh(mL) \]
\[ \therefore \epsilon_f = \frac{4 \times 30}{12 \times 10^{-3} \times 55} \tanh(24.72 \times 0.06) = 12.16 \]

ii. The fin efficiency
\[ \eta_f = \frac{\text{actual heat transfer rate from fin}}{\text{maximum possible heat transfer rate from fin}} \]
For a fin insulated at tip
\[ \eta_f = \frac{\sqrt{P h k A_c} (t_b - t_\infty) \tanh(mL)}{h (P L) (t_b - t_\infty)} = \frac{\tanh(mL)}{\sqrt{P h k A_c} L} \]
\[ \eta_f = \frac{\tanh(24.72 \times 0.06)}{24.72 \times 0.06} = 0.608 = 60.8\% \]

iii. Temperature at edge of the spine
Temperature distribution for insulated tip fin is given by
\[ t - t_a = \frac{\cosh (mL)}{\cosh (mL)} \frac{t_0 - t_a}{t_0 - t_a} \]
And temperature at the free end, \( x = L \)
\[ \frac{t_L - t_a}{t_0 - t_a} = \frac{1}{\cosh (mL)} \]
\[ \frac{t_L - 60}{100 - 60} = \frac{1}{\cosh(24.72 \times 0.06)} \]
\[ t_L = 60 + \frac{40}{\cosh(24.72 \times 0.06)} = 77.26^\circ C \]

iv. The heat dissipation with insulated tip fin
\[ Q_{fin} = k A_c (t_0 - t_a) \tanh(mL) \]
\[ Q_{fin} = 30 \times 1.13 \times 10^{-4} \times 24.72 \times (100 - 60) \times \tanh(24.72 \times 0.06) \]
\[ Q_{fin} = 3.023 W \]

Ex. 3.4.
A gas turbine blade made of stainless steel (\( k = 32 \text{ W/m-deg} \)) is 70 mm long, 500 mm\(^2\) cross sectional area and 120 mm perimeter. The temperature of the root of blade is 500\(^\circ\)C and it is exposed to the combustion product of the fuel passing from turbine at 830\(^\circ\)C. If the film coefficient between the blade and the combustion gases is 300 W/m\(^2\)-deg, determine:
(i) The temperature at the middle of blade,
(ii) The rate of heat flow from the blade.

Solution:
3. Heat Transfer from Extended Surface

Given data:

\( k = 32 \text{ W/m} - \text{deg}, \ L = 70 \text{ mm} = 0.07 \text{ m}, \ A_c = 500 \text{ mm}^2 = 500 \times 10^{-6} \text{ m}^2, \)
\( p = 120 \text{ mm} = 0.12 \text{ m}, \ t_0 = 500\text{°C}, \ t_a = 830\text{°C}, h = 300 \text{ W/m}^2 - \text{deg}, \)

\[
m = \sqrt{\frac{hp}{kA_c}} = \sqrt{\frac{300 \times 0.12}{32 \times 500 \times 10^{-6}}} = 47.43 \text{ m}^{-1}
\]

\[
mL = 47.43 \times 0.07 = 3.3201
\]

\[
h = \frac{300}{32 \times 47.43} = 0.1976
\]

\[
sinh(mL) = sinh(3.3201) = 13.81
\]

\[
cosh(mL) = cosh(3.3201) = 13.85
\]

\[
tanh(mL) = tanh(3.3201) = 0.997
\]

i. The temperature at the middle of blade

Temperature distribution for fin losing heat at the tip is given by

\[
t - t_a \over t_0 - t_a = \frac{\cosh m(L - x) + \frac{h}{km} \sinh m(L - x)}{\cosh(mL) + \frac{h}{km} \sinh(mL)}
\]

At the middle of the blade \( x = \frac{L}{2} = 0.035m \)

\[
cosh m(L - x) = cosh 47.43(0.07 - 0.035) = 2.725
\]

\[
sinh m(L - x) = sinh 47.43(0.07 - 0.035) = 2.534
\]

\[
t - 830 = \frac{2.725 + 0.1976 \times 2.534}{13.85 + 0.1976 \times 13.81} = \frac{3.226}{16.58} = 0.195
\]

\[
t = 830 + 0.195 \times (500 - 830) = 765.65\text{°C}
\]

ii. Heat flow through the blade is given by

\[
Q_{fin} = k A_c m(t_0 - t_a) \left[ \frac{\tanh(mL) + \frac{h}{km}}{1 + \frac{h}{km} \tanh(mL)} \right]
\]

\[
= 32 \times 500 \times 10^{-6} \times 47.43 \times (500 - 830) \left[ \frac{0.997 + 0.1976}{1 + 0.1976 \times 0.997} \right]
\]

\[
= -249.92 \text{ J}
\]

The –ve sign indicates that the heat flows from the combustion gases to the blade.

Ex. 3.5.

An electronic semiconductor device generates 0.16 kj/hr of heat. To keep the surface temperature at the upper safe limit of 75°C, it is desired that the generated heat should be dissipated to the surrounding environment which is at 30°C. The task is accomplished by attaching aluminum fins, 0.5 mm² square and 10 mm to the surface. Calculate the number of fins if thermal conductivity of fin material is 690 kj/m·hr·deg and the heat transfer coefficient is 45 kj/m²·hr·deg. Neglect the heat loss from the tip of the fin.

Solution:

**Given data:**

Prepared By: Mehul K. Pujara
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Page 3.22
Darshan Institute of Engineering & Technology, Rajkot
Heat Transfer

\[ Q_{total} = 0.16 \text{ kj/hr} = 0.044 \text{ W}, \ k = 690 \text{ kj/m} - \text{hr} - \text{deg} = 191.67 \text{ W/m} - \text{deg}, \]
\[ L = 10 \text{ mm} = 0.01 \text{ m}, \ A_c = 0.5 \text{ mm}^2 = 0.5 \times 10^{-6} \text{ m}^2, \ t_0 = 75\text{C}, \ t_a = 30\text{C}, \]
\[ h = 45 \text{ kj/m}^2 - \text{hr} - \text{deg} = 12.5 \text{ W/m}^2 - \text{deg}, \]

For square fin, \( A_c = b \times b = 0.5 \text{ mm}^2 \)
\[ \therefore b = \sqrt{0.5} = 0.70 \text{ mm} = 0.70 \times 10^{-3} \text{ m} \]

Perimeter of the fin is given by
\[ p = 4 \times b = 4 \times 0.70 \times 10^{-3} = 2.80 \times 10^{-3} \text{ m} \]
\[ m = \frac{hp}{kA_c} = \frac{12.5 \times 2.80 \times 10^{-3}}{191.67 \times 0.5 \times 10^{-6}} = 19.11 \text{ m}^{-1} \]
\[ mL = 19.11 \times 0.01 = 0.1911 \]

Heat loss from insulated tip fin is given by
\[ Q_{fin} = k A_c m(t_0 - t_a) \tanh(mL) \]
\[ Q_{fin} = 191.67 \times 0.5 \times 10^{-6} \times 0.1911 \times (75 - 30) \times \tanh 0.1911 \]
\[ Q_{fin} = 1.556 \times 10^{-4} \text{ W} \]

Total number of fins required are given by
\[ \text{no. of fins} = \frac{Q_{total}}{Q_{fin}} = \frac{0.044}{1.556 \times 10^{-4}} = 282.77 \]

So, to dissipate the required heat 283 no. of fins are required.

3.9 References

