UNIT-3 » RELATIONS, PARTIAL ORDERING AND RECURSION

PART-I RELATIONS

❖ INTRODUCTION
✓ Much of mathematics is about finding a pattern – a recognizable link between quantities that change. In our daily life, we come across many patterns that characterize relations such as brother and sister, father and son, teacher and student.

✓ In mathematics, also we come across many relations such as number $m$ is less than number $n$, line $l$ is parallel to line $m$, set $A$ is a subset of set $B$. In all these, we notice that a relation involves pairs of objects in certain order.

✓ In this chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair.

❖ RELATION
✓ Let $A$ and $B$ be non-empty sets. A binary relation or simply relation from $A$ to $B$ is a subset of $A \times B$.

✓ Suppose that $R$ is a relation from $A$ to $B$. Then, $R$ is a set of ordered pairs where each first element comes from $A$ and each second element comes from $B$. That is, for each pair, $x \in A$ and $y \in B$, exactly one of the following is true.

➢ $(x, y) \in R$, then we say $x$ is $R$-related to $y$, written as $x R y$.

➢ $(x, y) \notin R$, then we say $x$ is not $R$-related to $y$.

✓ Note: If $R$ is a relation from a set $A$ to itself, that is, if $R$ is a subset of $A^2 = A \times A$, then we say that $R$ is a relation on $A$.

❖ DOMAIN
✓ Let $R$ be a binary relation from $A$ to $B$. Then, the domain is denoted and defined as,

$$D_R = \{x \in A : (x, y) \in R\}$$

❖ RANGE
✓ Let $R$ be a binary relation from $A$ to $B$. Then, the range is denoted and defined as,

$$R_R = \{y \in B : (x, y) \in R\}$$
Examples

- A familiar example is relation “greater than” for real numbers. This relation is denoted by “ > ”. In fact, “ > ” should be considered as the name of a set whose elements are ordered pairs.

\[ \{(x, y) : x \text{ and } y \text{ are real numbers and } x > y\} \]

- The definition of relation permits any set of ordered pairs to define a relation like:

\[ S = \{(2,4), (1,3), (x, 6), (\text{veer,} *)\} \]

Here, \(D_r = \{2, 1, x, \text{veer}\} \) & \(R_r = \{4, 3, 6, *\} \)

**UNIVERSAL RELATION**

- Let \(A\) and \(B\) be two non-empty sets. Then, \(A \times B\), subset of itself, is called universal relation from \(A\) to \(B\).

- Example: Let \(A = \{1, 2\}\) and \(B = \{x, y\}\).

\[ A \times B = \{(1, x), (1, y), (2, x), (2, y)\} \]

**VOID RELATION**

- Let \(A\) and \(B\) be two non-empty sets. Then, the empty set \(\emptyset \subset A \times B\) is called void (null) relation from \(A\) to \(B\).

- Example: Let \(A = \{1, 2\}\) and \(B = \{x, y\}\). Then, the relation presented by the set \(\emptyset\) is called void (null) relation.

**UNION, INTERSECTION AND COMPLEMENT OPERATIONS ON RELATIONS**

- Let \(A = \{1, 2, 3, 4\}\) and

\[ R = \{(x, y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of 2}\} = \{(1, 3), (3, 1), (2, 4), (4, 2)\} \]

\[ S = \{(x, y) : \text{for } x, y \in A, x - y \text{ is an integral multiple of 3}\} = \{(1, 4), (4, 1)\} \]

- \(R \cup S = \{(1, 3), (3, 1), (2, 4), (4, 2), (1, 4), (4, 1)\} \)

- \(R \cap S = \emptyset \)

- For complement, let \(A = \{a, b\}\) and \(B = \{1, 2\}\), then \(A \times B = \{\{a, 1\}, \{b, 1\}, \{a, 2\}, \{b, 2\}\}\). Also, let \(R = \{\{a, 1\}, \{a, 2\}, \{b, 2\}\}\). Then, complement relation for \(R\) is \(\{\{b, 1\}\}\).
UNIT-3 ➤ RELATIONS, PARTIAL ORDERING AND RECURSION ➤

❖ PROPERTIES OF BINARY RELATIONS IN A SET

✓ Reflexive

➤ A binary relation R in a set A is said to be reflexive if, for every \( x \in A \),

\[ (x, x) \in R \]

✓ Irreflexive

➤ A binary relation R in a set A is said to be irreflexive if, for every \( x \in A \),

\[ (x, x) \notin R \]

✓ Symmetric

➤ A binary relation R in a set A is said to be symmetric if, for every \( x, y \in A \),

whenever \( (x, y) \in R \), then \( (y, x) \in R \)

✓ Antisymmetric

➤ A binary relation R in a set A is said to be antisymmetric if, for every \( x, y \in A \),

whenever \( (x, y) \in R \) & \( (y, x) \in R \), then \( x = y \)

✓ Transitive

➤ A binary relation R in a set A is said to be transitive if, for every \( x, y, z \in A \),

whenever \( (x, y) \in R \) & \( (y, z) \in R \), then \( (x, z) \in R \)

✓ Notes

➤ If relations R and S both are reflexive, then \( R \cup S \) and \( R \cap S \) are also reflexive.

➤ If relations R and S are symmetric and transitive, then \( R \cap S \) is also symmetric and transitive.

METHOD-1: BASIC EXAMPLES ON RELATION

<table>
<thead>
<tr>
<th>H</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Define the following terms with example.</td>
<td></td>
</tr>
</tbody>
</table>
| H 2 | Let \( A = \{(1,2), (2,4), (3,3)\} \) and \( B = \{(1,3), (2,4), (4,2)\} \).  
Find \( A \cup B, A \cap B, D(A), D(B), D(A \cup B), R(A), R(B) \) and \( R(A \cap B) \).  
**Answer:** \( A \cup B = \{(1,2), (2,4), (3,3), (1,3), (4,2)\}, A \cap B = \{(2,4)\}, D(A) = \{1,2,3\}, D(B) = \{1,2,4\}, D(A \cup B) = \{1,2,3,4\}, R(A) = \{2,4,3\}, R(B) = \{3,4,2\}, R(A \cap B) = \{4\} \). |
| C 3 | What are the ranges of the relations \( S = \{(x, x^2) : x \in \mathbb{N}\} \) and \( T = \{(x, 2x) : x \in \mathbb{N}\} \), where \( \mathbb{N} = \{0,1,2,3,...\} \)? Also, find \( S \cap T \) and \( S \cup T \).  
**Answer:** \( R(S) = \{x^2 : x \in \mathbb{N}\}, R(T) = \{2x : x \in \mathbb{N}\}, S \cap T = \{(0,0), (2,4)\}, S \cup T = \{(0,0), (1,1), (1,2), (2,4), (3,9), (3,6), (4,16), (4,8), ... ...\} \). |
| C 4 | Let \( L \) denotes the relation "less than or equal to" and \( D \) denotes the relation "divides", where \( x \) \( D \) \( y \) means "\( x \) divides \( y \)" defined on a set \( \{1,2,3,6\} \). Write \( L \) and \( D \) as a sets, and find \( L \cap D \).  
**Answer:** \( D = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\} \)  
\( L = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,3), (2,6), (3,3), (3,6), (6,6)\} \)  
\( L \cap D = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\} \) |
| H 5 | Give an example of a relation which is  
a) neither reflexive nor irreflexive?  
b) both symmetric and antisymmetric?  
c) reflexive but not symmetric?  
d) symmetric but not reflexive or transitive?  
e) transitive but not reflexive or symmetric?  
**Answer:**  
\( A = \{(1,1), (2,3), (3,2), (3,3)\}, B = \{(x, y) : x, y \in \mathbb{N}, x = y\}, C = \{(1,1), (1,2), (2,2)\}, D = \{(1,2), (2,1)\}, E = \{(1,2), (2,3), (1,3)\} \). |
| H 6 | Check whether the following relations are transitive or not.  
\( R_1 = \{(1,1)\}, R_2 = \{(1,2), (2,2)\}, R_3 = \{(1,2), (2,3), (1,3), (2,1)\} \)  
**Answer:** yes, yes, no |
| C 7 | Let \( L \) denotes the relation "less than or equal to" and \( D \) denotes the relation "divides", where \( x \) \( D \) \( y \) means "\( x \) divides \( y \)" defined on a set \( \{1,2,3,6\} \). Show that both \( L \) and \( D \) are reflexive, antisymmetric and transitive. |
Given $S = \{1,2,3,4,\ldots,10\}$ and a relation $R$ on $S$, where $R = \{(x,y) : x + y = 10\}$, what are the properties of the relation $R$?

Answer: symmetric but not reflexive, irreflexive, antisymmetric or transitive.

**RELATION MATRIX**

- A relation $R$ from a set $A$ to a set $B$ can be represented by a matrix called relation matrix of $R$.
- The relation matrix of $R$ can be represented by constructing a table whose columns are successive elements of $B$ & rows are successive elements of $A$. i.e. if $(x_i, y_j) \in R$, then we enter 1 in $i^{th}$ row and $j^{th}$ column. Similarly, if $(x_i, y_j) \notin R$, then we enter 0 in $i^{th}$ row and $j^{th}$ column.
- Let $A = \{x_1, x_2, x_3\}$ and $B = \{y_1, y_2\}$. Also, let $R = \{(x_1, y_1), (x_2, y_1), (x_3, y_2), (x_2, y_2)\}$. Then, the table representation looks like,

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence, the relation matrix is $M_R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

**GRAPH OF A RELATION**

- A relation can also be represented pictorially by drawing its graph.
- Let $R$ be a relation in $A = \{x_1, x_2, \ldots, x_m\}$. The elements of $A$ are represented by points or circles called nodes. The nodes may also be called vertices.
- Now, if $(x_i, x_j) \in R$, then we connect nodes $x_i$ and $x_j$ by an arc and put an arrow on the arc in the direction from $x_i$ to $x_j$. Thus, when all the nodes corresponding to the ordered pairs in $R$ are connected by arcs with proper arrows, we get a graph of the relation $R$.
- If $(x_i, x_j) \in R$ and $(x_j, x_i) \in R$, then we draw two arcs between $x_i$ and $x_j$, which is called a loop.
**UNIT-3 » RELATIONS, PARTIAL ORDERING AND RECURSION** [6]

### METHOD-2: EXAMPLES ON RELATION MATRIX AND GRAPH OF A RELATION

#### C 1
Let \( X = \{1,2,3,4\} \) and \( R = \{(x,y) : x > y\} \). Draw the graph of \( R \) and give its matrix.

**Answer:**
\[
M_R = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

#### H 2
Let \( A = \{a,b,c\} \) and denote the subsets of \( A \) by \( B_0, B_1, ..., B_7 \) given as \( B_0 = \emptyset, B_1 = \{c\}, B_2 = \{b\}, B_3 = \{b,c\}, B_4 = \{a\}, B_5 = \{a,c\}, B_6 = \{a,b\}, B_7 = \{a,b,c\} \). If \( R \) is the relation of proper subset on these subsets, then give the matrix of the relation.

**Answer:**
\[
M_R = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

#### H 3
Let \( X = \{1,2,3,4\} \) and \( R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\} \). Write matrix of \( R \) and sketch its graph.

**Answer:**
\[
M_R = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]
Determine the properties of the relations given by the graphs as given below. Also, write corresponding relation matrices.

Answer: relation (a) is antisymmetric, relation (b) is reflexive, relation (c) is reflexive and symmetric and relation (d) is transitive.

\[
M_a = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
M_b = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
M_c = \begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{bmatrix},
M_d = \begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

**Partition and Covering of a Set**

Let S be a given set and \( A = \{A_1, A_2, A_3, \ldots, A_m\} \) where each \( A_i; i = 1, 2, \ldots, m \) is a subset of S and
\[ \bigcup_{i=1}^{m} A_i = S \]

then, the set \(A\) is called a covering of \(S\) and the sets \(A_1, A_2, \ldots, A_m\) are said to be covers of \(S\).

\(\checkmark\) Also, if the elements of \(A\), which are subsets of \(S\), are mutually disjoint, then \(A\) is called a partition of \(S\) and sets \(A_1, A_2, \ldots, A_m\) are called the blocks of the partition.

\(\checkmark\) Example: Let \(S = \{x, y, z\}\). Then

- \(A = \{\{x, y\}, \{y, z\}\}\) is a covering of \(S\).
- \(B = \{\{x\}, \{y, z\}\}\) is a partition of \(S\).
- \(C = \{\{x, y, z\}\}\) is a partition of \(S\).
- \(D = \{\{x\}, \{y\}, \{z\}\}\) is a partition of \(S\).
- \(E = \{\{x\}, \{x, y\}, \{x, z\}\}\) is a covering of \(S\).

\(\checkmark\) Note: Every partition is a cover, but a cover may not be a partition.

\(\checkmark\) EQUIVALENCE RELATION

- A relation \(R\) on \(A\) is called an equivalence relation if it is reflexive, symmetric and transitive.

- Example: Let \(A = \{1, 2, \ldots, 7\}\) and \(R = \{(x, y) : x - y\) is divisible by 3\}.

  - For any \(a \in A\), \(a - a\) is divisible by 3. Hence, \(a R a\). So, \(R\) is reflexive.
  
  - For any \(a, b \in A\), let \(a R b\). i.e. \(a - b\) is divisible by 3. So, \(b - a\) is also divisible by 3. Hence, \(b R a\). So, \(R\) is symmetric.
  
  - For any \(a, b, c \in A\), let \(a R b\) and \(b R c\). i.e. \(a - b\) and \(b - c\) are divisible by 3. So, \(a - c = (a - b) + (b - c)\) is also divisible by 3. Hence, \(a R c\). So, \(R\) is transitive.

  - Hence, \(R\) is an equivalence relation on \(A\).

\(\checkmark\) EQUIVALENCE CLASS

- Let \(R\) be an equivalence relation on a set \(A\). For any \(x \in A\), the set \([x]_R \subseteq A\) given by

\[ [x]_R = \{y : y \in A \text{ and } x R y\} \]

is called an \(R\)-equivalence class generated by \(x \in A\).
**PROPERTIES OF EQUIVALENCE CLASS**

- For any $x \in A$, we have $x R x$ because $R$ is reflexive, therefore $x \in [x]_R$.
- Let $y \in A$ be any element such that $x R y$, then we have $[x]_R = [y]_R$.
- If $(x, y) \notin R$, then $[x]_R \neq [y]_R$. Because, if $[x]_R = [y]_R$, then there exist at least one $z \in [x]_R$ and $z \in [y]_R$, that gives $x R z$ and $y R z$, i.e. $x R y$, which contradicts to $(x, y) \notin R$.

**COMPATIBILITY RELATION**

- A relation $R$ in $A$ is said to be compatibility relation if it is reflexive and symmetric.
- Note: Every equivalence relation is a compatibility relation. But, reverse may not be true.

**MAXIMAL COMPATIBILITY BLOCK**

- Let $A$ be a set and $R$ be a compatibility relation on $A$. Then, a subset $C \subseteq A$ is called a maximal compatibility block if any element of $C$ is compatible to every other element of $C$ and no element of $A - C$ is compatible to all the elements of $C$.
- Let $R = \{(x, y) : x, y \in X$ and $x$ and $y$ contain some common letter$\}$ be a relation on $X = \{\text{ball, bed, dog, let, egg}$}. Then, $R$ is a compatibility relation denoted by "$\approx$". Also, note that $R$ is not equivalence relation. If we denote the elements of $X$ by $x_1, x_2, x_3, x_4, x_5$ then the graph is as shown here in figure (1).
✓ Since, relation is compatibility relation, it is not necessary to draw loops at each element nor it is necessary to draw both x R y and y R x. So, we can simplify graph as shown in figure (2).

✓ The relation matrix here is symmetric and has its diagonal elements unity. Therefore, it is sufficient to give only the elements of the lower triangular part only as shown as below.

\[
\begin{array}{c|cccc}
  & x_1 & x_2 & x_3 & x_4 \\
\hline
x_2 & 0 & & & \\
x_3 & 1 & 0 & & \\
x_4 & 1 & 1 & 0 & \\
x_5 & 0 & 1 & 1 & 1 \\
\end{array}
\]

✓ It is clear that the subsets \( \{x_1, x_2, x_4\} \), \( \{x_2, x_3, x_5\} \), \( \{x_2, x_4, x_5\} \) are maximal compatibility blocks.

❖ **HOW TO FIND MAXIMAL COMPATIBILITY BLOCK AND MATRIX**

✓ For this first we draw a simplified graph of the compatibility relation and pick from this graph the largest complete polygons i.e. a polygon in which any vertex is connected to every other vertex. In addition to this any element which is related only to itself forms a maximal compatibility block. Similarly, any two elements which are compatible to one another but to no other elements also form a maximal compatibility block. For example, triangle.

✓ Example: The maximal compatibility blocks of a compatibility relation R with simple graph as given below are \( \{1,3,4\} \), \( \{2,3\} \), \( \{4,5\} \), \( \{2,5\} \).

❖ Also, the matrix for this compatibility relation is as given below.

\[
\begin{array}{c|cccc}
  & x_1 & x_2 & x_3 & x_4 \\
\hline
x_2 & 0 & & & \\
x_3 & 1 & 0 & & \\
x_4 & 1 & 0 & 1 & \\
x_5 & 0 & 1 & 0 & 1 \\
\end{array}
\]
**COMPOSITE RELATION**

Let R be a relation from A to B and S be a relation from B to C. Then, a relation written as \( R \circ S \) is called a composite relation of R and S, defined by

\[
R \circ S = \{(x, z) : \text{for } x \in A \text{ and } z \in C \text{ there exist } y \in B \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}
\]

The operation of obtaining \( R \circ S \) is called composition of relations.

**CONVERSE OF A RELATION**

Given a relation R from A to B, a relation \( R^\sim \) from B to A is called the converse of R, where the ordered pairs of \( R^\sim \) are obtained by interchanging the members in each of the ordered pairs of R. This means, for \( x \in A \) and \( y \in B \), \( x R y \iff y R^\sim x \).

**TRANSITIVE CLOSURE OF A RELATION**

Let R be relation in a finite set A. The relation \( R = R \cup R^2 \cup R^3 \cup \ldots \) in A is called the transitive closure of R in A.

**METHOD-3: EXAMPLES ON EQUIVALENCE AND COMPATIBILITY RELATION**

|---|---|
| C | Let \( X = \{1, 2, ..., 7\} \) and \( R = \{ (x, y) : x - y \text{ is divisible by } 3 \} \). Show that R is an equivalence relation. Also, draw the graph of R.  
**Answer:** |
| H | Let \( X = \{1, 2, ..., 7\} \) and \( R = \{ (x, y) : x - y \text{ is even} \} \). Show that R is an equivalence relation. |
Let \( Z = A_1 \cup A_2 \cup A_3 \).
Where \( A_1 = \{…, 1, 4, 7, …\}, A_2 = \{…, 2, 5, 8, …\} \) and \( A_3 = \{…, 3, 6, 9, …\} \).

Then, define equivalence relation whose equivalence classes are \( A_1, A_2, A_3 \).

**Answer:** \( R = \{(x, y) : \text{x - y is divisible by 3} \} \) over the set of integers

Let \( z \) be the set of integers and \( R \) be the relation called "congruence modulo 3" defined by \( R = \{(x, y) : x - y \text{ is divisible by 3} \} \). Determine equivalence classes generated by the elements of \( z \).

**Answer:** \( z/R = \{[0]_R,[1]_R,[2]_R\} \)

Prove that relation "congruence modulo \( m \)" given by \( R = \{(x, y) : x - y \text{ is divisible by } m \} \), over the set of positive integers, is an equivalence relation.

Let \( S \) be the set of all statement functions in \( n \) variables and let \( R \) be the relation given by \( R = \{(x, y) : x \leftrightarrow y \} \). Discuss the equivalence classes generated by the elements of \( S \).

**Answer:** there are \( 2^2^n \) \( R \)-equivalence classes

Let \( X = \{a, b, c, d, e\} \) and let \( C = \{\{a, b\}, \{c\}, \{d, e\}\} \). Show that the partition \( C \) defines an equivalence relation on \( X \).

Let \( R \) denote a relation on the set of ordered pairs of positive integers such that \( (x, y) R (u, v) \) iff \( xv = yu \). Show that \( R \) is an equivalence relation.

Let \( R = \{(x, y) : x, y \in X \text{ and } x \text{ and } y \text{ contain some common letter} \} \) be relation on \( X \) where \( X = \{\text{ball, bed, dog, let, egg}\} \). Then, show that \( R \) is a compatibility relation.
Let the compatibility relation on a set \([x_1, x_2, x_3, \ldots, x_6]\) be given by the following matrix. Draw the graph. Also, find maximal compatibility blocks of the relation.

\[
\begin{array}{cccccc}
& x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
x_1 & 1 & 0 & 1 & 0 & 1 \\
x_2 & 0 & 1 & 1 & 0 & 1 \\
x_3 & 0 & 0 & 1 & 1 & 0 \\
x_4 & 0 & 0 & 0 & 1 & 0 \\
x_5 & 0 & 0 & 0 & 0 & 1 \\
x_6 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Answer: \([x_1, x_2, x_3], [x_1, x_3, x_6], [x_3, x_4, x_5], [x_3, x_5, x_6]\)

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Let \(R = \{(1, 2), (3, 4), (2, 2)\}\) and \(S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}\). Find \(R \circ S, S \circ R, R \circ (S \circ R), (R \circ S) \circ R, R \circ R, S \circ S,\) and \(R \circ R \circ R\).

Answer: \(R \circ S = \{(1, 5), (3, 2), (2, 5)\}\), \(S \circ R = \{(4, 2), (3, 2), (1, 4)\}\)

\[R \circ R = \{(1, 2), (2, 2)\}\]

\[S \circ S = \{(4, 5), (3, 3), (1, 1)\}\]

\[R \circ (S \circ R) = \{(1, 2), (2, 2)\}\]

\[(R \circ S) \circ R = (R \circ S) \circ (R \circ R) = \{(3, 2)\}\]

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Let \(R = \{(1, 2), (3, 4), (2, 2)\}\) and \(S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}\) be relations on a set \(A = \{1, 2, 3, 4, 5\}\). Obtain relation matrices for \(R \circ S\) and \(S \circ R\).

Answer: \(M_{R \circ S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\)

\(M_{S \circ R} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\)
### Unit-3 Relations, Partial Ordering and Recursion [14]

**H 15**

Given the relation matrix \( M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \) of a relation \( R \) on a set \( \{a, b, c\} \), find the relation matrices of \( \bar{R}, R^2 = R \circ R, R^3 = R \circ R \circ R, \) and \( R \circ \bar{R} \).

**Answer:**

\[
M_{\bar{R}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad M_{R^2} = M_{R^3} = M_{R \circ \bar{R}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\]

**C 16**

Given the relation matrices \( M_R \) and \( M_S \), find \( M_{R \circ S}, M_{\bar{R}}, M_{\bar{S}}, M_{R \circ \bar{S}}, \) and show that \( M_{R \circ \bar{S}} = M_{\bar{S}} \circ R \).

\[
M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}
\]

**Answer:**

\[
M_{R \circ S} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad M_{\bar{R}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad M_{\bar{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

\[
M_{R \circ \bar{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]